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Harnessing Lagrange Multipliers as Optimality Condition for Solving Nonlinear Constrained Optimization Problems

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Abstract- This research work succinctly investigated the solution of Nonlinear Optimization Problems using Lagrange Multiplier as optimality condition. The historical development of nonlinear optimization was discussed. The nonlinear constrained optimization problems with equality constraints were solved with the method of Lagrange multiplier.

I. INTRODUCTION

Optimization is the act of obtaining the best result under given circumstances. It is also finding solutions from set of admissible or feasible solutions that minimizes (or maximizes) a performance measure or objective, Asim Karim (2003). The technique allows comparison of the different choices for determining which decision might be best. The goal of all such decisions is either to minimize the effort required or maximize the desired benefit. Since the effort required or the benefit desired in any practical situation can be expressed as a function of certain decision variables, optimization can be defined as the process of finding the conditions that give the maximum or minimum value of a function, f(x). According to Shang (1997), optimization problems are made up of three components: a set of unknowns (variables), an objective function to be minimized or maximized and a set of constraints that specify feasible values of the variables. These sets of unknown variables are called non-negativity constraints or decision variables.

In Xue Jin (2021), the direct idea is to use mathematical tools such as differential calculus, variational method and Lagrange multiplier method to obtain the solution expression of the problem through logical derivation and analysis. Hence, we succinctly investigated classical optimization methods which are analytical in nature and make use of differential calculus to find optimal value for both unconstrained and constrained objective functions. We shall devote more time on the constrained optimization problems with equality and inequality constraints.

According to Freund (2014), most of the theoretical and computational interest in nonlinear optimization has taken place since 1947. However, it is useful to note that nonlinear optimization was first studied as early as the 1600s. Indeed, both Fermat and Newton studied the 1-dimensional nonlinear optimization problem:

$$max\ f(x)\ where\ x\ \in\ R$$

and Newton developed the classical optimality condition:

$$\frac{df(x)}{dx} = 0$$

This was generalized by Euler to multivariable nonlinear optimization:

$$min\ f(x_1, \ldots, x_n);$$

with the optimality condition:

$$\nabla f(x)\ =\ 0$$

According to Huijuan Li, (2008), the Lagrange multiplier method is a basic mathematical tool for constrained optimization of differentiable functions, especially for nonlinear constrained optimization. He discussed its application in the field of power system economic operation. Huijuan said that optimization problems seek to minimize or maximize a real function which plays an important in the real world. This can be classified into unconstrained optimization problems and constrained optimization problems. Many practical uses in science, engineering, economics or even in our everyday life can be formulated as constrained optimization problems,

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such as the minimization of energy of a particle in physics; how to maximize the profit of investment in economics. (Mikulas Luptacik 2010)

Rockafellar, (1993), stated that Lagrange multipliers are viewed as auxiliary variables introduced in a problem of constrained minimization to write first-order optimality conditions formally as a system equation. He further said that equality constraints $f_i(x) = 0$ and inequality constraints $f_i(x) \le 0$ are most common in describing feasible solutions, but other side conditions, like the attainability of x as a state taken on by a controlled dynamical system, are encountered too. Such further conditions can be indicated abstractly by a requirement $x \, \varepsilon \, X$ with $X \subset R^n$

In a work done in optimal design problems, Gavi and Scruggs, (2020) said that if inequality $g(x) \le 0$ constrains the minimum of $f(x)$, then the optimum point of the augmented objective function $J_A(x, \lambda) = f(x) + \lambda g(x)$ is minimized with respect to x and maximized with respect to λ. The optimization problem

$$\underset{x_1, x_2, \ldots, x_n}{Min\, J} = f(x_1, x_2, \ldots, x_n)$$

$$g_1(x_1, x_2, \ldots, x_n) \le 0$$

$$g_2(x_1, x_2, \ldots, x_n) \le 0$$

$$\vdots$$

$$g_m(x_1, x_2, \ldots, x_n) \le 0$$

may be written in terms of augmented objective function

$$\underset{\lambda_1, \lambda_2, \ldots \lambda_m}{Max} \; \underset{x_1, x_2, \ldots, x_n}{Min} J_A(x_1, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots \lambda_m)$$

such that $\lambda_i \ge 0 \; \forall \, j$ the necessary conditions for optimality are:

$$\frac{\partial J_A}{\partial x_k} \Big|_{\substack{x_i = x_i^* \\ \lambda_j = \lambda_j^*}} = 0$$

$$\frac{\partial J_A}{\partial \lambda_k} \Big|_{\substack{x_i = x_i^* \\ \lambda_j = \lambda_j^*}} = 0$$

$$\lambda_j \ge 0$$

The equations define the relations used to derive expressions for, or compute values of, the optimal design variables x_i^*. Lagrange multipliers for inequality constraints $g_j(x_1, x_2, \ldots, x_n) \le 0$, are non-negative. If an inequality $g_j(x_1, x_2, \ldots, x_n) \le 0$ constrains the optimum point, the corresponding Lagrange multiplier, λ_j, is positive. If an inequality $g_j(x_1, x_2, \ldots, x_n) \le 0$ does not constrain the optimum point, the corresponding Lagrange multiplier λ_j is set to zero.

II. PRELIMINARIES

2.1 Multivariable Optimization with Equality Constraints

According to Agu et al (2024), to optimize a multivariable function, we use the concept of partial derivative because they measure the change in the dependent variable due to unit change in one of the independent variables.

As we discuss the problem of optimizing a continuous and differentiable function subject to equality constraints. That is,

$Optimize \qquad Z = f(x_1, x_2, \ldots, x_n)$
(2.1)
subject to
$g_i = (x_1, x_2, \ldots, x_n) = k_i, \quad i = I(1)m$

There are various methods for solving the above defined problems, but our concentration is only o Lagrange.

According to Clive Newstead (2014), Lagrange multipliers give us a means of optimizing multivariate functions subject to several constraints on their variables. Clive further stated the following theorems and examples of multivariable optimization with different numbers of variables and constraints.

Theorem 2.1.1: If $(x, y) = (a, b)$ is a solution to the problem of maximization $f(x, y)$ subject to the constraint $g(x, y) = k$, then there is a scalar λ such that $\nabla f(a, b) = \lambda \nabla g(a, b)$. We call λ a Lagrange multiplier, which represent the amount of change in the objective function due to the percentage change in the constraint limit.

The Lagrangian of the problem of maximizing $f(x, y)$ subject to $g(x, y) = k$ is the function of $n + 1$ variables defined by

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$L(x, y; \lambda) = f(x, y) + \lambda[k - g(x, y)]$ (2.2)

Working with the Lagrangian gives us a systematic way of finding optimal values.

Theorem 2.1.2: If $(x, y) = (a, b)$ is a solution to the problem of maximizing $f(x, y)$ subject to the constraint $g(x, y) = k$. and L is the Lagrangian, then there is some scalar λ such that $\nabla L(a, b; \lambda) = 0$ and $\det[H_L(a, b; \lambda)] > 0$

Thus, a procedure for finding the maximum value of $f(x, y)$ subject to $g(x, y) = k$ can thus be found by carrying out the following steps:

Step 1: Solve the system equation given $\nabla L = 0$, where L is the Lagrangian.
Step 2: For each value $(a, b; \lambda)$ obtained from Step 1, check whether $\det[H_L(a, b; \lambda)] > 0$. Then $(a, b; \lambda)$ is a solution to the problem. The solution from Step 2 for which $f(a, b)$ is the greatest is the solution to the maximization problem.

Notice that all the above is for maximization, not minimization. However, a function f can be minimized by maximizing its negative, $-f$. Thus, the Lagrangian for the problem of minimizing $f(x, y)$ subject to $g(x, y) = k$ is

$L(x, y; \lambda) = -f(x, y) + \lambda[k - g(x, y)]$ (2.3)
And the procedure for finding a solution to the minimization problem is identical from this point on.

2.2 Optimality Conditions
A necessary condition for minimizers is that the function's slope is zero at the minimum, this is according to Scott Moura (2014). Consider the first-order necessary conditions for optimality for Nonlinear Optimization problems. These conditions provide a set of nonlinear equations that can be solved to determine the optimal solution under certain assumptions.

Consider the equality constrained optimization problem

$$\min f(x)$$

subject to

$$h_j(x) = 0, \qquad j = 1, \dots, l$$

Introduced the so-called Lagrange multipliers, $\lambda_j, j = 1, .., l$. Then we can augment the cost (objective) function to form the "Lagrangian", $L(x)$ as follows

$$L(x) = f(x) + \sum_{j=1}^{l} \lambda_i h_j(x)$$
$$= f(x) + \lambda^T h(x)$$

Note that when all constraints are satisfied, that is $h(x) = 0$, then the second term becomes zero.

Consequently, the Lagrangian $L(x)$ and the cost function $f(x)$ provide identical values for all feasible x. We now state the first-order necessary condition (FONC) for equality constrained problems.

Preposition 2.2.1: (FONC for Equality Constrained Nonlinear Optimization Problem).
If a local minimum x^* exist, then it satisfies

$\frac{\partial L}{\partial \lambda}(x^*) = \frac{\partial f}{\partial x}(x^*) + \lambda^T \frac{\partial h}{\partial x}(x^*) = 0 \qquad (Stationality)$ (2.4)
$\frac{\partial L}{\partial \lambda}(x^*) = h(x^*) = 0 \qquad (feasibility)$ (2.5)

That is, the gradient of the Lagrangian is zero at the minimum x^*
Remark: This condition is only necessary. That is, if a local minimum x^* exists, then it must satisfy the FONC. However, a design x which satisfies the FONC isn't necessary a local minimum.

Remark: If the optimization problem is convex, then the FONC is necessary and sufficient. That is, a design x which satisfies the FONC is also a local minimum.
Example: Consider the equality constrained quadratic programming

$\min \frac{1}{2} x^T Q\, x + R^T x$ (2.6)
subject to
$Ax = b$ (2.7)
From the Lagrangian
$L(x) = \frac{1}{2} x^T Q x + R^T x + \lambda^T (Ax - b)$ (2.8)
Then the FONC is
$\frac{\partial L}{\partial \lambda}(x^*) = Qx^* + R + A^T \lambda = 0$ (2.9)

Combing the FONC with the equality constraint yields

$$\begin{bmatrix} Q & A^T \\ A & 0 \end{bmatrix}\begin{bmatrix} x^* \\ \lambda \end{bmatrix} = \begin{bmatrix} -R \\ b \end{bmatrix}$$

which provides a set of linear equations that can be solved directly.

III. MATHEMATICAL PROOF FOR LAGRANGE MULTIPLIERS METHODS

We look at nonlinear optimization problem which has four variables and two equality constraints. According to Huijuan Li (2008) and Ejikeme et al (2015), we have,

$$Optimize\ u = f(x, y, z, t) \qquad (3.1)$$

subject to:

$$\Phi(x, y, z, t) = 0 \qquad (3.2)$$

$$\Psi(x, y, z, t) = 0 \qquad (3.3)$$

Assuming that at the point $P(a, b, c, d)$ the function takes an extreme value when compared with the values at the neighbouring points satisfying the constraints. Also, if the point P, the Jacobian

$$\frac{\partial(\Phi,\Psi)}{\partial(z,t)} = \Phi_z\Psi_t - \Phi_t\Psi_z \neq 0 \qquad (3.4)$$

Thus, at the point of P, we can represent two of the variables, say z and t, as functions of the other two x and y, by using $eqn(3.2)$ and $eqn\ (3.3)$. Hence, substitute the functions $z = g(x, y)$ and $t = h(x, y)$ in the objective function, $eqn\ (3.1)$, then we get an objective function of two independent variables x and y, and this function must have a free extreme value at the point $x = a, y = b$ that is its two partial derivatives must vanish at the point. The two following equations must therefore hold.

$$f_x + f_z\frac{\partial z}{\partial x} + f_t\frac{\partial t}{\partial x} = 0 \qquad (3.5)$$

$$f_y + f_z\frac{\partial z}{\partial x} + f_t\frac{\partial t}{\partial x} = 0 \qquad (3.6)$$

We can determine two numbers λ and μ in such a way that the two equations

$$f_z - \lambda\Phi_z - \mu\Psi_z = 0 \qquad (3.7)$$

$$f_t - \lambda\Phi_t - \mu\Psi_t = 0 \qquad (3.8)$$

are satisfied at the point where the extreme value occurs. Take the partial derivative with respect to x on $eqn\ (3.2)$ and $eqn\ (3.3)$ and get

$$\Phi_x + \Phi_z\frac{\partial z}{\partial x} + \Phi_t\frac{\partial t}{\partial x} = 0 \ (3.9)$$

$$\Psi_x + \Psi_z\frac{\partial z}{\partial x} + \Psi_t\frac{\partial t}{\partial x} = 0 \qquad (3.10)$$

Multiply $eqn\ (3.9)$ and $eqn\ (3.10)$ by $-\lambda$ and $-\mu$ respectively and add them to $eqn\ (3.5)$ and we get

$$f_x - \lambda\Phi_x - \mu\Psi_x + (f_z - \lambda\Phi_z - \mu\Psi_z)\frac{\partial z}{\partial x} + (f_t - \lambda\Phi_t - \mu\Psi_t)\frac{\partial t}{\partial x} = 0 \qquad (3.11)$$

Hence by definition of λ and μ, we can get

$$f_x - \lambda\Phi_x - \mu\Psi_x = 0 \qquad (3.12)$$

Similarly, we can get

$$f_y - \lambda\Phi_y - \mu\Psi_y = 0 \qquad (3.13)$$

Thus, we arrive at the same conclusion as shown in $eqn\ (7) \Rightarrow$

$$\nabla f = 0$$

or $eqn\ (8) \Rightarrow$

$$\frac{\partial f}{\partial x_i} - \sum_{m=1}^{n}\lambda_m\frac{\partial G_m}{\partial x_i} = 0\,, i = 1, \ldots, n$$

and $eqn\ (9) \Rightarrow$

$$G(x_1, \ldots, x_n) = 0$$

The proving method can be similarly extended to the objective functions with more variables and more constraints.

3.1 Constrained Multivariable Optimization with Equality Constraints

Let us look at the problem of optimizing a continuous and differentiable function subject to equality constraints, that is:

$$Optimize\ (Max\ or\ Min)\ Z = f(x_1, x_2, \ldots, x_n) \qquad (3.14)$$

subject to the constraints

$$h_i(x_1, x_2, \ldots, x_n) = b_i; i = 1, 2, \ldots, m \qquad (3.15)$$

Here it is assumed that $m < n$ to get the solution. There are various methods for solving the above defined problem. We shall look at Direct Substitution Method.

3.2 Direct Substitution Method

The constraint set is continuous and differentiable therefore any variable in the constraint set can be expressed in terms of the remaining variables, Benjamin et al (1999). Then it shall be substituted into the objective function. The new objective function, so obtained, is not subject to any constraint and hence its optimum value can be obtained by the unconstrained optimization.

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Example 3.2.1: find the optimum solution of the following constrained multi-variable problem.

Minimize $Z = x_1^2 + (x_2 + 1)^2 + (x_3 - 1)^2$

subject to the constraint

$x_1 + 5x_2 - 3x_3 = 6$ and

$x_1; x_2, x_3 \geq 0$

Solution: Since the given problem has three variables and one equality constraint, any one of the variables can be removed from Z with the help of the equality constraints. Let us choose variable x_3 to be eliminated from Z . Then, from the equality constraint, we have

$$x_3 = \frac{x_1 + 5x_2 - 6}{3}$$

Substitute the value of x_3 in the objective function

$$Z = x_1^2 + (x_2 + 1)^2 + (x_3 - 1)^2$$

$$f(x) = Z = x_1^2 + (x_2 + 1)^2 + \left(\frac{x_1 + 5x_2 - 6}{3} - 1\right)^2$$

$$= x_1^2 + (x_2 + 1)^2 + \left(\frac{x_1 + 5x_2 - 6 - 3}{3}\right)^2$$

$$x_1^2 + (x_2 + 1)^2 + \frac{1}{9}(x_1 + 5x_2 - 9)$$

The necessary condition for minimum of Z is that the gradient

$$\nabla f(x) = \left[\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}\right] = 0$$

That is,

$$\frac{\partial Z}{\partial x_1} = 2x_1 + \frac{2}{9}(x_1 + 5x_2 - 9) = 0$$

$$\frac{\partial Z}{\partial x_2} = 2(x_2 + 1) + \frac{10}{9}(x_1 + 5x_2 - 9) = 0$$

$$\Rightarrow \quad 18x_1 + 2x_1 + 10x_2 - 18 = 0$$

$$20x_1 + 10x_2 = 18$$

$$10x_1 + 5x_2 = 9 \ldots\ldots\ldots\ldots\ldots\ldots(i)$$

and

$$18x_2 + 18 + 10x_1 + 50x_2 - 90 = 0$$

$$\Rightarrow \quad 10x_1 + 68x_2 = 72$$

$$5x_1 + 34x_2 = 36 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(ii)$$

$$(i) \times 1: \; 10x_1 + 5x_2 = 9 \ldots\ldots\ldots\ldots(iii)$$

$$(ii) \times 2: \; 10x_1 + 68x_2 = 72 \ldots\ldots\ldots(iv)$$

substract (iv) from (iii);

$$\Rightarrow \quad -63x_2 = -63$$

$x_2 = 1$

substituted 1 for x_2 into (i)

$$\Rightarrow \quad 10x_1 + 5(1) = 9$$

$$\Rightarrow \quad 10x_1 = 4$$

$$x_1 = \frac{4}{10} = \frac{2}{5}$$

$$x_1 = \frac{2}{15}, \qquad x_2 = 1$$

To find whether the solution, so obtained, is minimum or not, we must apply the sufficient condition by forming Hessian matrix. The Hessian matrix for the given objective function is

$$H(x_1, x_2) = \begin{bmatrix} \frac{\partial^2 Z}{\partial x_1^2} & \frac{\partial^2 Z}{\partial x_1 \partial x_2} \\ \frac{\partial^2 Z}{\partial x_2 \partial x_1} & \frac{\partial^2 Z}{\partial x_2^2} \end{bmatrix}$$

$$= \begin{bmatrix} 2 + \frac{2}{9} & \frac{10}{9} \\ \frac{10}{9} & 2 + \frac{50}{9} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{20}{9} & \frac{10}{9} \\ \frac{10}{9} & \frac{68}{9} \end{bmatrix}$$

Let's show that the above matrix is a positive definite:

$$\begin{bmatrix} \frac{20}{9} - \lambda & \frac{10}{9} \\ \frac{10}{9} & \frac{68}{9} - \lambda \end{bmatrix} = 0$$

$$\Leftrightarrow \quad \left(\frac{20}{9} - \lambda\right)\left(\frac{68}{9} - \lambda\right) - \frac{100}{81} = 0$$

$$\Leftrightarrow \quad \frac{1360}{81} - \frac{88}{9}\lambda + \lambda^2 - \frac{100}{81} = 0$$

$$\Leftrightarrow \quad \lambda^2 - \frac{88}{9}\lambda + -\frac{1260}{81} = 0$$

$$\Leftrightarrow \quad \left(\lambda - \frac{18}{9}\right)\left(\lambda - \frac{70}{9}\right) = 0 \Rightarrow \lambda = 2 \; or \; \frac{70}{9}$$

Since the matrix is symmetric and principal diagonal elements are positive, $H(x_1, x_2)$ is positive definite and the objective function is convex. Hence, the optimum solution to the given problem is

$$x_1 = \frac{2}{5}, \; x_2 = 1, \; x_3 = \frac{-1}{5} \; and \; \min Z = \frac{28}{5}$$

4 Using Lagrange multipliers method to solve optimization problems

Consider the following prototype problems:

Problem 4.1: A company produces steel widgets which are formed into little trinkets that people love to buy. The main cost incurred is the cost of labour and cost of steel. It has been determined that the hourly cost of labour is #200 while one tone of steel costs #20,000. It has also been determined by an analyst that the company faces a revenue function of the form,

$$R(h, s) = 100h^{\frac{2}{3}}s^{\frac{1}{3}} \tag{4.1}$$

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where h is the number of hours worked and s is the number of tons of steel used within one production period. The company has a budget of #200,000 per production period. What is the maximum revenue the company can expect to make per production period? The problem of the company can be cast mathematically as follows:

$$Maximize\ R(h,s) = 100h^{\frac{2}{3}}s^{\frac{1}{3}}$$

subject to

$$200h + 20{,}000s = 200{,}000$$

Or

$$h + 100s = 1{,}000$$

since the company has a budget of #200,000 per production period, the expenditure cannot exceed this amount. Thus, the budget constraint becomes:

$$200h + 20{,}000s \leq 200{,}000$$

Or

$$h + 100s \leq 1{,}000$$

the inequality is appreciated from the reality that expenditure normally does not exceed budget but could be less than or equal to it. Let's give the constraint name say,

$$g(h,s) = 20h + 2000s - 20000 \qquad (4.2)$$

Consider the set (h,s) of all possible inputs maybe with h the number of hours of labour, on one axis and s, the number of tons of steel purchased, on the other axis. The constraint denotes a half-plane (the graph of a linear function). The revenue function is a one-parameter family of contours, $R(h,s) = r$. Different values of r produce different contours. However, the objective is to identify the value that touches the constraint curve. That is, tangent to it at a given point, because that will be the contour line where if you move the value by just a little, it would no longer intersect with the curve, there would no longer be valves for h and s that satisfy the constraint. Hence, the best way to think about finding that tangency is to consider the vector perpendicular to the tangent line to the curve at that point.

The figure below is the illustration:

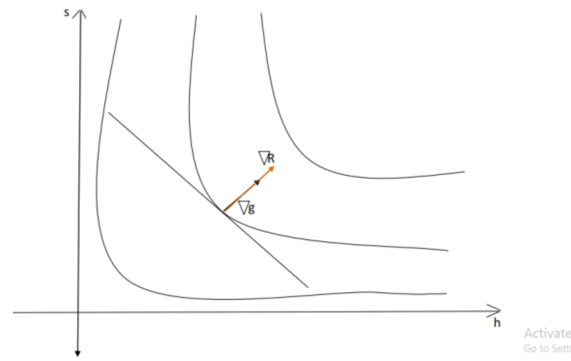

Figure 4.1

∇R, represented by the gradient, the gradient of our revenue R function. i.e., the function whose contour is the revenue. Tangent to the constraint line is that there will be another vector, the gradient of g, of our constraint function, that points in the same direction, that is proportional to the revenue R. That is to say that the gradient of the revenue function is proportional to the gradient of the constraint g, i.e.,

$$\nabla R = \lambda \nabla g \qquad (4.3)$$

and λ is the proportionality constant which is called Lagrange multiplier. Considering the above eqn (4,3). Let's compute the gradient of R. Therefore, the gradient of R is column vector of the partial derivative of R with respect to its first variable, h and the second component is the partial derivative with respect to second variable, s.

$$\nabla R = \begin{bmatrix} \frac{\partial R}{\partial h} \\ \frac{\partial R}{\partial s} \end{bmatrix} = \begin{bmatrix} 100 \times \frac{2}{3} h^{-\frac{1}{3}} s^{\frac{1}{3}} \\ 100 \times \frac{2}{3} h^{\frac{2}{3}} s^{-\frac{2}{3}} \end{bmatrix}$$

Take partial derivative of g with respect to h and s

$$\nabla g = \begin{bmatrix} \frac{\partial g}{\partial h} \\ \frac{\partial g}{\partial s} \end{bmatrix} = \begin{bmatrix} 20 \\ 2000 \end{bmatrix}$$

then

$$\frac{200}{3} \frac{s^{\frac{1}{3}}}{h^{\frac{1}{3}}} = 20\lambda$$

and

$$\frac{100}{3} \frac{h^{\frac{2}{3}}}{s^{\frac{2}{3}}} = 2000\lambda$$

let $u = \frac{s}{h}$

$$\frac{200}{3} u^{\frac{1}{3}} = 20\lambda \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (i)$$

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$\frac{100}{3}u^{-\frac{2}{3}} = 2000\lambda$ ………………………………………….(ii)

$(i) \times \frac{3}{200}: u^{\frac{1}{3}}$

$= \frac{3}{10}\lambda$ …………………………………….(iii)

$(ii) \times \frac{3}{100}: u^{-\frac{2}{3}}$

$= 60\lambda$ …………………………………….(iv)

Multiply both (iii) and (iv) by $u^{\frac{2}{3}}$

$\Rightarrow u = \frac{3}{10}\lambda u^{\frac{2}{3}}$ ………………………………..(v)

and

$1 = 60\lambda u^{\frac{2}{3}}$ ………………………………………….(vi)

$(v) \times 200 : 200u$

$= 60\lambda u^{\frac{2}{3}}$ ……………………………..(vii)

$\Rightarrow (vi) = (vii)$ then,

$$1 = 200u$$

$$\Rightarrow\ 200\left(\frac{s}{h}\right) = 1$$

$$\Rightarrow\ \mathrm{h = 200s}$$

Since $200h + 2000s = 20000$

$\Rightarrow 20(200s) + 2000s = 20000$

$$4000s + 2000s = 20000$$

$$6000s = 20000$$

$$s = \frac{20000}{6000} = \frac{10}{3}$$

and

$$h = 200 \times \frac{10}{3} = \frac{2000}{3}$$

Recall we are maximizing

$$R(h,s) = 100h^{\frac{2}{3}}s^{\frac{1}{3}}$$

Therefore, the maximum revenue is

$$R\left(\frac{2000}{3}, \frac{10}{3}\right) = 100\left(\frac{2000}{3}\right)^{\frac{2}{3}}\left(\frac{10}{3}\right)^{\frac{1}{3}} = N11{,}396.08$$

Problem 4.2: The temperature of a point (x, y) on a circular metal plate is given

$$T(x,y) = Kxy \quad (K > 0\ \ Constant)$$

What are the points of maximum and minimum temperature on a plate and their temperature if the plate has a radius of r.

Solution: Let the center of the metal plate be the point (g, h). Then, the plate is given by

$$(x-h)^2 + (y-g)^2 \le r^2$$

so the problem becomes

$$Optimize\ T(x,y) = Kxy$$

subject to:

$$(x-h)^2 + (y-g)^2 \le r^2$$

Introduce complementary slack,

$$(x-h)^2 + (y-g)^2 + s^2 = r^2$$

Construct the Lagrangian

$$L(x,y,\lambda,s) + \lambda(r^2 - (x-h)^2 - (y-g)^2 - s^2)$$

$\frac{\partial L}{\partial x} = 0 \Rightarrow ky - 2\lambda(x-h) = 0$ ………………(i)

$\frac{\partial L}{\partial \lambda} = 0 \Rightarrow (x-h)^2 - (y-g)^2 + s^2 = r^2$ ………………..(ii)

$\frac{\partial L}{\partial y} = 0 \Rightarrow Kx - 2\lambda(y-g) = 0$ ……………………………(iii)

$\frac{\partial L}{\partial s} = 0 \Rightarrow -2\lambda s = 0$ ……………………………………..(iv)

$$(i) \times y : Ky^2 - 2\lambda xy + 2\lambda hy = 0$$

$$(ii) \times x : Kx^2 - 2\lambda xy + 2\lambda gx = 0$$

$$ky^2 + 2\lambda hy = Kx^2 + 2\lambda gx$$

$$y^2 + 2\lambda\frac{h}{K}y = x^2 + 2\lambda\frac{r}{K}x$$

$$\left(y + \lambda\frac{h}{K}\right)^2 - \lambda^2\frac{h^2}{K^2} = (x + \lambda\frac{g}{K})^2 - \lambda^2\frac{g^2}{K^2}$$

To simplifying matters assume, by translation of axis if need be, that the center of the plate is $(0,0)$, i.e, $h = 0 = g$. Then,

(a). $ky^2 = kx^2 \ \Leftrightarrow\ y^2 = x^2 \ \Leftrightarrow y = \pm x$

(b). $x^2 + y^2 + s^2 = r^2$ so $2x^2 = r^2 \Rightarrow x = \pm\frac{r}{\sqrt{2}}$

(c). $s \le 0, s = 0$

Problem 4.3: The temperature at a point (x, y) on a metal plate in the xy −plane is given by

$$T(x,y) = 6xy$$

Determine the maximum temperature on the circle

$$x^2 + y^2 = 8$$

Solution

$$Maxmize\ T(x,y) = 6xy$$

subject to

$$g(x,y) = x^2 + y^2 - 8$$

we solve the Lagrangian

$$\nabla T = \lambda\nabla g$$

and

$$g(x,y) = 0$$

Consider

$$L = T - \lambda$$

$$= 6xy - \lambda(x^2 + y^2 - 8)$$

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$$\Rightarrow L_x = 0 = 6x - 2\lambda x \ldots\ldots\ldots\ldots\ldots\ldots\ldots (i)$$
$$\Rightarrow L_y = 0 = 6x - 2\lambda y \ldots\ldots\ldots\ldots\ldots\ldots\ldots (ii)$$

and

$$g = 0 = x^2 + y^2 - 8 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (iii)$$

then,

$$(i) \times y: 0 = 6y^2 - 2\lambda xy \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (iv)$$
$$(ii) \times x: 0 = 6x^2 - 2\lambda xy \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (v)$$

equate (iv) and (v)

$$\Rightarrow 6y^2 - 2\lambda xy = 6x^2 - 2\lambda xy$$
$$6y^2 = 6x^2$$
$$y^2 = x^2$$

substitute x^2 for y^2 into (iii) there we have

$$0 = x^2 + y^2 - 8$$
$$\Rightarrow \quad 0 = x^2 + x^2 - 8$$
$$2x^2 = 8$$
$$x^2 = 4$$
$$x = \pm 2$$

$\Rightarrow \quad y = \pm 2$ (since $x = y$)

Hence we have the following points $(2, 2); (2, -2); (-2, 2); (-2, -2)$. let's test for our points of interest. Recall that

$$T(x, y) = 6xy$$
$$\Rightarrow \quad T(2, 2) = 24$$
$$T(-2, 2) = 24$$
$$T(2, -2) = 24$$
$$T(-2, -2) = 24$$

Therefore, the maximum value of temperature T is $24°C$ at the points $(2, 2)$ and $(-2, -2)$

Problem 4.4: Use Lagrange multiplier to find the extrema of the following problems. Assume x and y to be positive.

(a) Maximize $f(x, y) = x^2 + y^2$ subject to $x + y = 4$
(b) Maximize $f(x, y) = e^{xy}$ subject to $x^2 + y^2 = 8$

Solution (a)

$$f(x, y) = x^2 + y^2$$
$$g(x, y) = x + y - 4$$

Using Lagrangian

$$L(x, y; \lambda) = f(x, y) - \lambda g(x, y)$$
$$\Rightarrow \quad L(x, y; \lambda) = x^2 + y^2 - \lambda(x + y - 4)$$

then,

$$L_x = 2x - \lambda = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (i)$$
$$L_y = 2y - \lambda = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (ii)$$
$$g = x + y - 4 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (iii)$$

equate (i) and (ii)

$$\Rightarrow \quad 2x - \lambda = 2y - \lambda$$
$$\Rightarrow \quad x = y$$

substitute x for y into (iii) we have,

$$x + y - 4 = 0$$
$$\Rightarrow x + x = 4$$
$$2x = 4$$
$$x = 2 \quad \Rightarrow \quad y = 2$$

from (i)

$$\lambda = 2x$$
$$\Rightarrow \quad \lambda = 2(2) = 4$$

then $f(2, 2) = 8$ is the maximum value at the point $(2, 2)$

Check: Take $y = 1$, then $x + y = 4 \Rightarrow x + 1 = 4 \Rightarrow x = 3$.

$$f(3, 1) = 3^2 + 1 = 10$$
$$\Rightarrow \quad 8 < 10$$

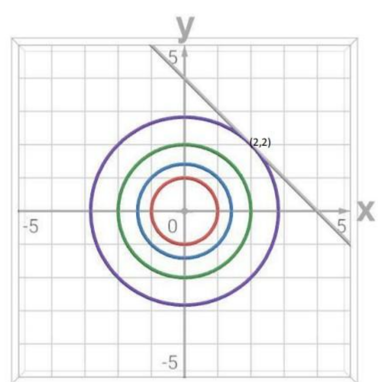

Figure 4.2

Solution (b)

$$f(x, y) = e^{xy}$$
$$f(x, y) = x^2 + y^2 - 8$$
$$f_x = ye^{xy}$$
$$f_y = xe^{xy}$$

and

$$g_x = 2x$$
$$g_y = 2y$$

using

$$\Delta f = \nabla g$$
$$\Rightarrow \quad ye^{xy} = 2\lambda x \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (i)$$
$$xe^{xy} = 2\lambda y \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (ii)$$
$$x^2 + y^2 = 8 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (iii)$$

then,

$$(i) \times y: \; y^2 e^{xy} = 2xy\lambda$$
$$(ii) \times x: \; x^2 e^{xy} = 2xy\lambda$$
$$\Rightarrow \; x^2 = y^2$$

substitute x^2 for y^2 into (iii)

$x^2 + y^2 = 8$

$x^2 + x^2 = 8$

$2x^2 = 8$

$$x^2 = 4$$
$$x = \pm 2 \quad \Rightarrow \quad y = \pm 2$$

from (i)

$$\lambda = \frac{ye^{xy}}{2x}$$

at $(2, 2)$ $\quad \lambda = \frac{2e^{2\times2}}{2\times2} = \frac{e^4}{2}$

at $(2, -2)$ $\quad \lambda = \frac{-2e^{2\times2}}{2\times2} = \frac{-e^4}{2}$

Hence, we have the following points $(2, 2); (2, -2); (-2, 2); (-2, -2)$. We can now test for our points of interest.

$$f(2, 2) = e^{2\times2} = e^4$$
$$f(2, -2) = e^{2\times-2} = e^{-4} < e^4$$
$$f(-2, 2) = e^{-2\times2} = e^{-4} < e^4$$
$$f(-2, -2) = e^{-2\times-2} = e^4$$

therefore, the maximum value of $f(x, y) = e^{xy}$ is e^4 at $(2, 2)$ and $(-2, -2)$

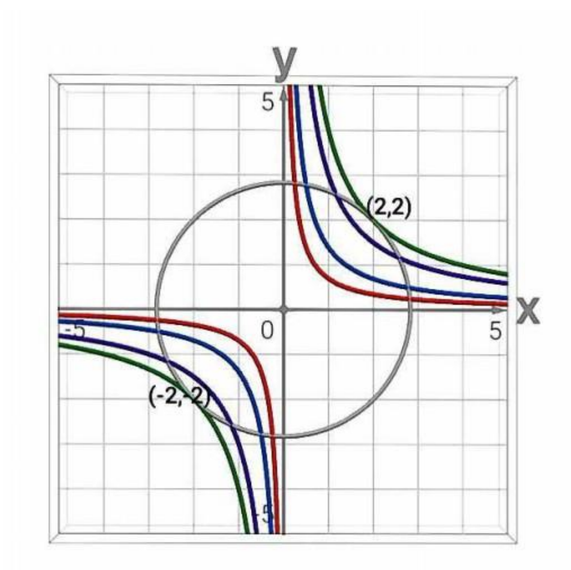

Figure 4.

V. SUMMARY, CONCLUSION AND RECOMMENDATION

5.1 Summary

In this research work, it has been established that the method of Lagrange Multiplier is used in making best choice in economics decisions and other vital areas of life. The nonlinear constraint optimization problem with equality constraints are solved with the method of Lagrange Multipliers.

The nonlinear program must include at least one nonlinear function, which could be the objective function or some or all the constraints.

5.2 Conclusion

It is very important to note that all the aims and objectives of this research work have been reasonably achieved. We have seen that it is necessary to consider optimization approach when it comes to profit maximization and cost minimization. Moreover, Lagrange Multipliers is a useful mathematical tools in dealing with economic and other issues of life

5.3 Recommendation

Operation Research is a real life problem solving area of Mathematics, therefore, it should be expected that students are exposed early to this fundamentals in their journey of studying Mathematics and other related courses. More empirical researches in this area should be encouraged with the use of mathematical software

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