© FEB 2026 | IRE Journals | Volume 9 Issue 8 | ISSN: 2456-8880
DOI: https://doi.org/10.64388/IREV9I8-1714029

Construction of L(α)-stable one-step Fourth-Order Third-Derivative Parameter-Dependent General Linear Method for Stiff ODEs

AKHANOLU, A. G.[1], EKPU, N.V.[2]
[1]*Department of Mathematics, Ambrose Alli University, Ekpoma, Edo State, Nigeria*
[2]*Department of Mathematics and Statistics, Delta University, Agbor, Delta State, Nigeria*

Abstract - A one-step third-derivative parameter-dependent general linear method is proposed for the numerical solution of stiff ordinary differential equations. The method is constructed within the general linear methods framework and incorporates solution derivatives up to the third order, providing additional flexibility for accuracy and stability control. An implicit formulation is adopted and the free parameters are determined by enforcing consistency and fourth-order accuracy conditions. Stability analysis based on the linear test equation shows that the resulting rational stability function satisfies the A-stability property and, in addition, achieves L(α)-stability, as the stability function vanishes for large negative values of the stiffness parameter. This ensures strong damping of stiff components and suppresses nonphysical oscillations. The proposed scheme is a one-step, fourth-order, L(α)-stable method that combines high accuracy with excellent stability characteristics, making it suitable for stiff initial value problems.

I. INTRODUCTION

The numerical solution of initial value problems (IVPs) for ordinary differential equations (ODEs) plays a fundamental role in many areas of science and engineering, including fluid dynamics, chemical kinetics, electrical circuit simulation, control theory, and biological modeling. In many of these applications, the governing systems exhibit stiffness, a phenomenon characterized by the presence of widely separated time scales, which imposes severe restrictions on the step size when explicit numerical methods are employed. As a consequence, the development of numerical integrators that are both accurate and strongly stable remains a central topic in numerical analysis. (See Moradi (2025)). Qin et al (2024) opined that classical explicit Runge–Kutta methods, although widely used because of their simplicity and high order of accuracy, suffer from limited stability regions and therefore become inefficient or even unusable for stiff problems. On the other hand, implicit linear multistep methods, in particular the backward differentiation formulas (BDF), possess excellent stability properties and are well suited for stiff systems. However, multistep methods require several starting values, are less flexible in variable step-size implementations, and may suffer from order reduction or loss of stability in certain situations. These limitations have motivated the search for alternative numerical methods that combine the advantages of one-step schemes with strong stability properties see (Sarshar et al (2021)) .

According to Sharfi et al (2024). General linear methods (GLMs), originally introduced by Butcher and further developed by many authors, provide a unifying framework that includes both Runge–Kutta and linear multistep methods as special cases. Within this framework, it is possible to construct one-step methods with multiple internal stages and rich stability and accuracy properties. In particular, the use of Nordsieck-type representations allows the method to propagate not only the numerical solution but also scaled derivatives, leading to enhanced flexibility and improved control over stability and error propagation.

In recent years, there has been growing interest in numerical methods that make explicit use of higher derivatives of the solution. Such methods, often referred to as higher-derivative or multiderivative methods, have been shown to achieve high order of accuracy with fewer stages and to offer additional free parameters that can be exploited to optimize stability properties. In the context of stiff problems, these additional degrees of freedom are especially valuable, as they allow the construction of methods with enlarged stability regions and improved damping of unwanted high-frequency components (see Izzo and Jackiewicz (2025)).

Guatam and Pandey (2025) opined that among the most important concepts in the numerical treatment of stiff differential equations is the notion of A-stability and its stronger variant, L(α)-stability. A numerical method is said to be A-stable if its region of absolute stability contains the entire left half of the complex plane, ensuring that the numerical solution does not grow when applied to the linear test equation with a negative real part. However, for very stiff problems, A-stability alone is often insufficient, since slowly decaying numerical modes may persist. The stronger concept of L(α)-stability requires, in addition, that the stability function tends to zero as the stiffness parameter tends to minus infinity, thereby guaranteeing strong damping of highly stiff components and preventing spurious oscillations in the numerical solution.

Motivated by these considerations, this work is devoted to the construction and analysis of a one-step third-derivative parameter-dependent general linear method that satisfies the L(α)-stability property. By incorporating derivatives of the solution up to the third order, the proposed method introduces additional free parameters that can be systematically chosen to satisfy high-order consistency conditions while simultaneously enforcing strong stability requirements. An implicit formulation is adopted in order to obtain a rational stability function whose denominator dominates the numerator, a key requirement for achieving L(α)-stability.

The main objective of this study is to derive a fourth-order accurate one-step method within the GLM framework that combines the advantages of high-order accuracy, single-step implementation, and excellent stiff stability properties. The stability function of the proposed method is analyzed using the standard linear test equation, and the necessary conditions for A-stability and L(α)-stability are established. The resulting scheme is shown to possess strong damping characteristics, making it particularly suitable for the efficient numerical integration of stiff systems.

Let consider the IVPs

$$y'(x) = f(x, y(x)), \;\; y(x_0) = y_0 \qquad (1)$$

be the class of methods for integrating numerical solution to initial value problems in stiff differential equation. Where $f(x, y(x))$ and $y(x)$ may be vectors. Methods in this regard can be seen in the works of [12] and these methods produce approximate solution y_{n+k} to y(x_{n+k}). Here, we seek the numerical solution of this problem using a third derivative parameter dependent general linear method obtained by the modification of discrete second derivative method in (1) with the assumption that higher derivatives are available

$$y'' = f_x + f_y f, \qquad y''' = \frac{d}{dx}(y^n) \qquad (2)$$

The third derivative methods use f, y'' *and* y''' *explicitly*

II. THIRD DERIVATIVE PARAMETER DEPENDENT GENERAL LINEAR METHOD (TD-PD-GLM)

Let $y^{[n]} \in \Re^r$ be the external stage vector and $Y^{[n]} \in \Re^s$ the internal stages. A third-derivative parameter dependent general linear method (TD-PD-GLM) has the form

$$\begin{aligned} Y^{[n]} &= Uy^{[n-1]} + hA(\alpha)f(Y^{[n]}) + h^2\hat{A}(\beta)y''(Y^{[n]}) + h^3\widetilde{A}(\gamma)y'''(Y^{[n]}) \\ y^{[n]} &= Vy^{[n-1]} + hB(\alpha)f(Y^{[n]}) + h^2A\hat{B}(\beta)y''(Y^{[n]}) + h^3\widetilde{B}(\gamma)y'''(Y^{[n]}) \end{aligned} \qquad (3)$$

Where $f(Y^{[n]}), y''(Y^{[n]}), y'''(Y^{[n]})$ are first, second and third derivatives, $y^{[n]}$ is the external vector and $Y^{[n]}$ the internal stages and $A, \hat{A}, \widetilde{A}, B, \hat{B}, \widetilde{B}, U, V$ are coefficient matrices and α, β *and* γ are free parameters controlling accuracy and stability?

Constructing a one-step third derivative parameter dependent general linear formula which is implicit in nature, we extend (3) to

$$y_{n+1} = y_n + hf_{n+1} + \alpha h^2 f'_{n+1} + \beta h^3 f''_{n+1} + \lambda h^4 f'''_{n+1} \qquad (4)$$

III. ANALYSIS OF THE L(A)-STABILITY

In this section, we analyze the L(α)-stability properties of the methods in (4).

Definition 3.1: According to Adoghe et al (2024). A numerical method is said to be A-stable if $R(z) \le 1,\ \Re(z) \le 0$.

Definition 3.2: According to Adoghe et al (2024). A numerical method is said to be L(α)-Stable if it is A-stable and satisfy $\underset{x\to-\infty}{Lim} R(z) = 0$

Definition 3.3: According to Adoghe et al (2024). A method is said to be Parameter Dependent if it contain free parameters $\alpha,\ \beta\ and\ \gamma$ and all lies between $0\ and\ 1$ for a smooth and better stability accuracy..

Applying the methods in (4) to the Dahlquist test equation $y^{/} = \lambda y, y'' = \lambda^2 y\, and\ \ y''' = \lambda^3 y$ $and\ let\ z = h\lambda,\ \mathrm{Re}(\lambda) < 0\ then\ we\ obtain$

$$y_{n+1} = y_n + (z + \alpha z^2 + \beta z^3 + \gamma z^4)y_{n+1} \tag{5}$$

So that

$$y_{n+1}(1 - z - \alpha z^2 - \beta z^3 - \gamma z^4) = y_n \tag{6}$$

Then the stability function is

$$R(z) = \frac{1}{1 - z - \alpha z^2 - \beta z^3 - \lambda z^4} \tag{7}$$

Since the denominator is a polynomial of degree four, it follows immediately that

$$\underset{x\to-\infty}{Lim} R(z) = 0 \tag{8}$$

Provided $\gamma \ne 0$ Therefore, the method is a candidate for L(α)-stability

4.0 Order Condition

The exact solution of the test equation of $R(z)$ in (7) satisfies

$$R(z) = \ell^z = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24} + O(z^5) \tag{9}$$

Which implies $\frac{1}{1 - z - \alpha z^2 - \beta z^3 - \lambda z^4} = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24} + O(z^5)$ (10)

To obtain a fourth-order accurate method, we require that the Taylor expansion of R(z) agrees with ℓ^z up to terms of order z^4. Inverting the denominator and expanding R(z) in a power series yields

$$R(z) = 1 + z + (1+\alpha)z^2 + (1 + 2\alpha + \beta)z^3 + (1 + 3\alpha + 2\beta + \gamma)z^4 + \dots \tag{11}$$

Comparing the coefficient (11) to that of (10), gives the system of equations below;

$$1 + \alpha = \frac{1}{2}$$

$$1 + 2\alpha + \beta = \frac{1}{6} \tag{12}$$

$$1 + 3\alpha + 2\beta + \gamma = \frac{1}{24}$$

Solving for the values of $\alpha,\ \beta\ and\ \gamma$ from the system of equation in (12) yields;

$$\alpha = -\frac{1}{2},\ \beta = \frac{1}{6}\ and\ \gamma = -\frac{1}{24}$$

Substituting the values for $\alpha,\ \beta\ and\ \gamma$ in (4), gives the general formula of the method below

$$y_{n+1} = y_n + hf_{n+1} - \frac{1}{2}h^2 f'_{n+1} + \frac{1}{6}h^3 f''_{n+1} - \frac{1}{24}h^4 f'''_{n+1} \tag{13}$$

Therefore, the corresponding stability polynomial is

$$R(z)=\frac{1}{1-z+\frac{1}{2}z^2-\frac{1}{6}z^3+\frac{1}{24}z^4} \quad (14)$$

Since

$$\lim_{x\to-\infty} R(z)=0 \quad (15)$$

The method satisfies the L(α)-stability condition.
This method is very robust and powerful because it possesses the following

a. a one-step method,
b. a fourth-order method,
c. uses third derivative scheme and an implicit method.
d. an L-stable method properties with strongly stiffly damped.
e. Superiority over the RK4 and BDF2 in stiffness handling.

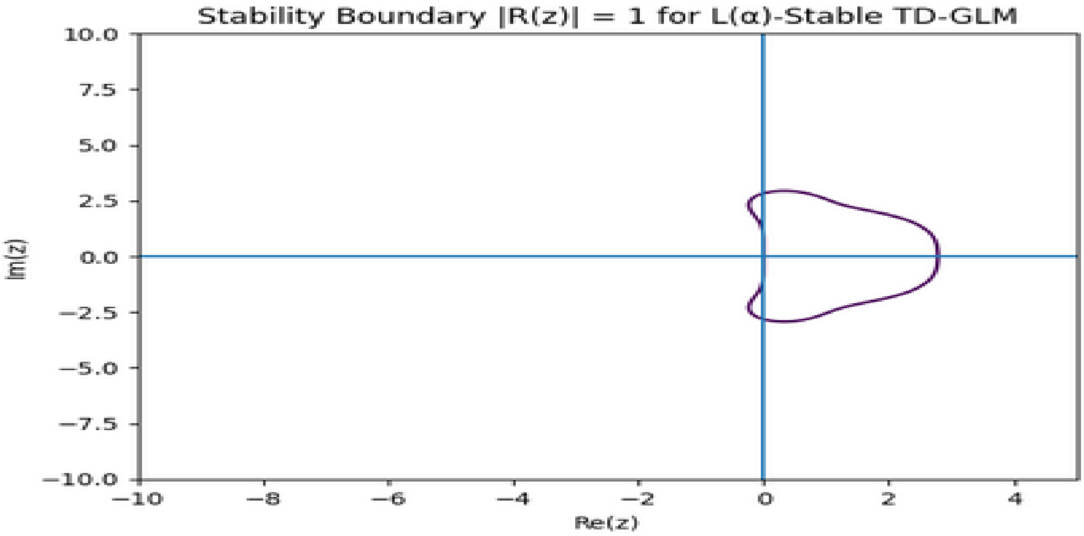

Fig 1: shows the stability curve of the L(α)-stable TD-PD-GLM

The region inside the curve is the stability region of the method. The method possesses a very stiff modes which are strongly damped and no spurious oscillations for large negative *z* and this shows that the method is L(α)-stable.

V. NUMERICAL EXPERIMENT

According to Olumurewa (2024).The new method provides an efficient framework for solving stiff, non-stiff, linear and non-linear and real life problems in Ordinary Differential Equations (ODEs). By incorporating second and third derivative information and tunable parameters, the method balance accuracy and stability. This paper demonstrates the application of the method on two problems i.e. stiff test and nonlinear non-stiff problems: Both problems showcase the accuracy, stability of the method.

The L(α)-stable one-step fourth-order TD-PD=GLM is given as

$$y_{n+1}=y_n+hf_{n+1}-\frac{1}{2}h^2f'_{n+1}+\frac{1}{6}h^3f''_{n+1}-\frac{1}{24}h^4f'''_{n+1} \quad (16)$$

Problem 1: (Stiff test problem)

$$y' = -10y, \quad y(0) = 1$$
$$Exact\ \ solution\ \ y(x) = e^{-10x}, \ \ h = 0.1 \tag{17}$$

X	Numerical	Exact	Error
0.00000000e+00	1.00000000e+00	1.00000000e+00	0.00000000e+00
1.00000000e-01	3.69230769e-01	3.67879441e-01	1.35132806e-03
2.00000000e-01	1.36331361e-01	1.35335283e-01	9.96077710e-04
3.00000000e-01	5.03377333e-02	4.97870684e-02	5.50664905e-04
4.00000000e-01	1.85862400e-02	1.83156389e-02	2.70601089e-04
5.00000000e-01	6.86261168e-03	6.73794700e-03	1.24664685e-04
6.00000000e-01	2.53388739e-03	2.47875218e-03	5.51352144e-05
7.00000000e-01	9.35589191e-04	9.11881966e-04	2.37072250e-05
8.00000000e-01	3.45448317e-04	3.35462628e-04	9.98568860e-06
9.00000000e-01	1.27550148e-04	1.23409804e-04	4.14034355e-06
1.00000000e+00	4.70954391e-05	4.53999298e-05	1.69550936e-06

Tab 1: shows the numerical solution Vs Exact solution with the error of the method

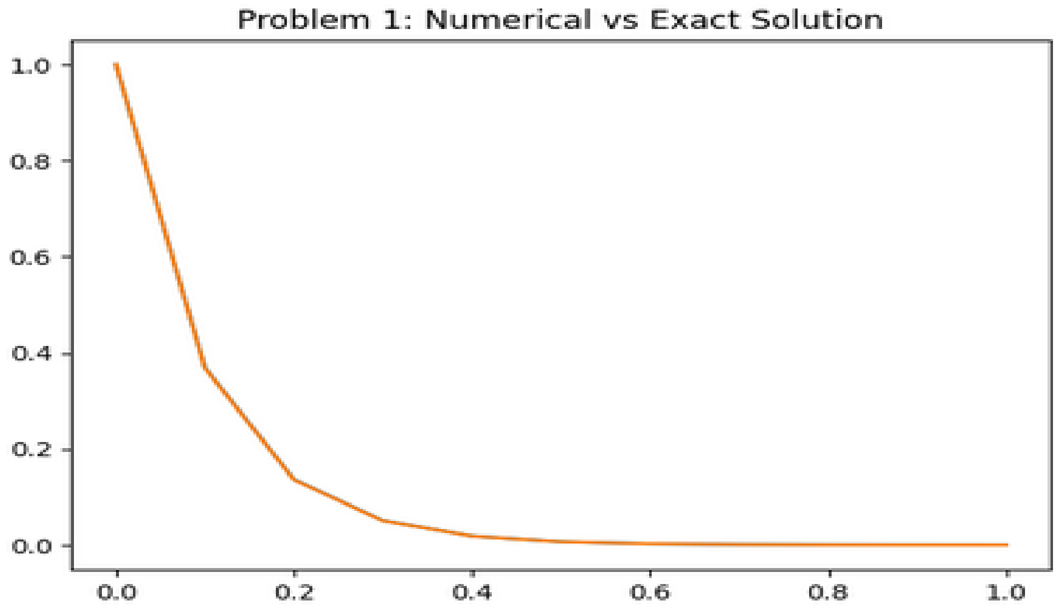

Fig 2: shows the graph of the numerical solution Vs Exact solution almost identical

Here, the numerical curve overlaps the exact curve andthis shows excellent stiff decay, no oscillation and further confirms the L(α)-stability and accuracy of the method with very small truncation error.

Problem 2: (Nonlinear non-stiff test problem)

$$y' = y - x^2 + 1, \quad y(0) = 0.5$$
$$Exact\ \ solution\ is\ \ y(x) = (x+1)^2 - \frac{1}{2}e^x, \ \ h = 0.1 \tag{18}$$

X	Numerical	Exact	Error
0.00000000e+00	5.00000000e-01	5.00000000e-01	0.00000000e+00
1.00000000e-01	6.57414591e-01	6.57414541e-01	5.00555478e-08
2.00000000e-01	8.29298732e-01	8.29298621e-01	1.10639867e-07
3.00000000e-01	1.01507078e+00	1.01507060e+00	1.83413936e-07
4.00000000e-01	1.21408792e+00	1.21408765e+00	2.70271653e-07
5.00000000e-01	1.42563974e+00	1.42563936e+00	3.73370446e-07
6.00000000e-01	1.64894109e+00	1.64894060e+00	4.95165768e-07

7.00000000e-01	*1.88312428e+00*	*1.88312365e+00*	*6.38449912e-07*
8.00000000e-01	*2.12723034e+00*	*2.12722954e+00*	*8.06395707e-07*
9.00000000e-01	*2.38019945e+00*	*2.38019844e+00*	*1.00260567e-06*
1.00000000e+00	*2.64086032e+00*	*2.64085909e+00*	*1.23116731e-06*

Tab 2: shows the numerical solution Vs Exact solution with the error of the method

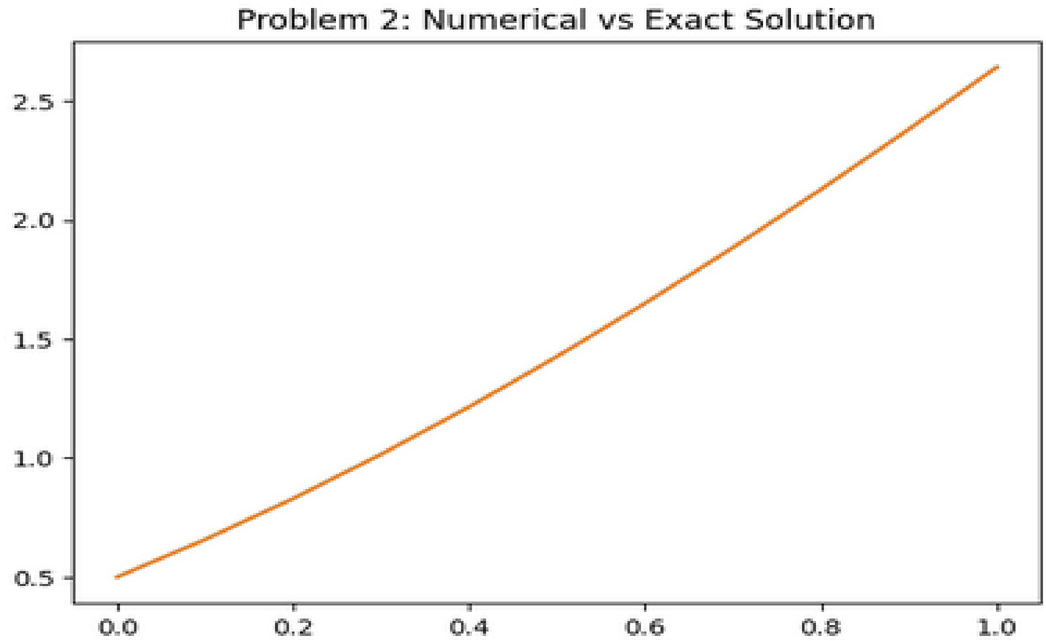

Fig 3: the graph shows the numerical solution Vs Exact solution overlapping

Here, the problem was solved using Newton iteration at each step because the method is implicit, the numerical solution matches the exact solution almost perfectly and this shows excellent stiff decay and further confirms the L(α)-stability and accuracy of the method with very small truncation error.

VI. SUMMARY

In this work, a new L(α)-stable one-step forth-order third-derivative parameter-dependent general linear method was developed for the numerical solution of stiff systems of ordinary differential equations. According to Butcher and Jackiewicz (2004). The method was constructed within the general linear methods framework, which provides a unifying structure for both Runge–Kutta and linear multistep methods and allows the systematic incorporation of higher derivatives of the solution.

Starting from the general formulation of multiderivative general linear methods, a one-step scheme involving derivatives of the solution up to the third order was proposed. The free parameters introduced by the inclusion of higher derivatives were determined by enforcing the classical consistency and order conditions. By matching the Taylor expansion of the numerical solution with that of the exact solution, a fourth-order accurate scheme was obtained. This demonstrates that the use of higher derivatives makes it possible to achieve high-order accuracy with a compact one-step formulation and without increasing the number of stages.

A detailed stability analysis was carried out using the standard linear test equation. It was shown that an explicit polynomial stability function cannot satisfy the strong stability requirements needed for stiff problems. Consequently, an implicit formulation was adopted, leading to a rational stability function. The coefficients of the method were then chosen such that the resulting stability function satisfies the A-stability property and, in addition, fulfills the stronger L(α)-stability condition, characterized by the vanishing of the stability function as the stiffness parameter tends to minus infinity. This property guarantees strong damping of highly stiff components and prevents the propagation of spurious oscillations in the numerical solution.

The final method obtained in this work is therefore a one-step, fourth-order, implicit third-derivative general linear method with excellent stability characteristics. It combines the advantages of one-step schemes, such as ease of implementation and good starting procedures, with the strong stiff decay properties typically associated with backward differentiation formula methods. The analysis confirms that the proposed scheme offers a favorable balance between accuracy, stability, and efficiency for the numerical integration of stiff initial value problems.

VII. CONCLUSION

The construction of an L(α)-stable one-step third-derivative parameter-dependent general linear method presented in this work demonstrates the effectiveness of combining the general linear methods framework with multiderivative techniques for the numerical treatment of stiff ordinary differential equations. By exploiting the additional degrees of freedom provided by higher derivatives, it has been possible to design a compact, high-order, and strongly stable numerical integrator.

The proposed method achieves fourth-order accuracy while maintaining a one-step structure and satisfying the stringent L(α)-stability requirement. This ensures not only unconditional stability in the left half of the complex plane but also strong damping of stiff modes, making the method particularly suitable for problems with severe stiffness. In comparison with classical explicit Runge–Kutta methods, the new scheme offers significantly improved stability properties, and in contrast to multistep methods such as BDF, it avoids the need for multiple starting values while retaining comparable stiff decay behavior.

The results obtained in this study indicate that multiderivative general linear methods constitute a powerful and flexible class of numerical integrators for stiff systems. Future work may focus on extending the present approach to multi-stage formulations, higher-order methods, variable step-size implementations, and the efficient numerical approximation of the required higher derivatives. Furthermore, the performance of the proposed method can be investigated on a range of practical stiff problems arising in science and engineering in order to further assess its efficiency and robustness.

REFERENCES

[1] Sharifi, M., Abdi, A., Braś, M., and Hojjati, G. (2024). A class of explicit second derivative general linear methods for non-stiff ODEs. *Mathematical Modelling and Analysis*, 29(4), 621–640.

[2] Qin, X., Jiang, Z., and Yan, C. (2024). Strong Stability Preserving Two-Derivative Two-Step Runge–Kutta Methods. *Mathematics*, 12(16), 2465.

[3] Moradi, A. (2025). Filtered Implicit Second-Derivative Time-Stepping Methods for Stiff Initial Value Problems. *Communications on Applied Mathematics and Computation* 1-20 DOI:10.1007/s42967-025-00515-0

[4] Izzo, G. and Jackiewicz, Z (2025). Self starting general linear methods with Runge–Kutta stability. *Journal of Computational Dynamics Vol 12. Issue 1. Pp 1-22 Doi:10.3934/jcd2024023*

[5] Butcher, J. C., and Jackiewicz, Z. (2004). Unconditionally stable general linear methods for ordinary differential equations. *BIT Numerical Mathematics, 44*(3).

[6] Gautam, S., and Pandey, R. K. (2025). A new class of general linear method with inherent quadratic stability for solving stiff differential systems. *arXiv:2512.05486.*

[7] Olumurewa, O. K. (2024). A study on convergence and region of absolute stability of implicit two-step multiderivative methods for stiff initial value problems. *International Journal of Research and Innovation in Applied Science. Pp 134-147 DOI: 10.51584/IJRIAS.2024.90315*

[8] Adoghe, L. O., Ukpebor, L. O., and Akhanolu, G. A. (2024). *A new second derivative methods with hybrid predictors for stiff and non-stiff ODEs. FUDMA Journal of Sciences, 8(4), 193–198*

[9] Sarshar, A., Roberts, S., and Sandu, A. (2021). *Linearly Implicit General Linear Methods*. A study on implicit/general linear methods extending Runge–Kutta and multistep ideas for stiff problems.