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An Improved Inertial Viscosity Tseng-Type Method for Pseudomonotone variational Inequalities in Hilbert Spaces

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Abstract- In this paper, we propose a single inertial viscosity Tseng-type algorithm for solving the variational inequality and fixed-point problem in real Hilbert spaces. The method integrates inertial extrapolation, viscosity approximation, and a Mann-type iterative scheme with an adaptive step-size strategy. The cost operator is assumed to be pseudomonotone and Lipschitz continuous, while the fixed-point mapping is quasi-nonexpansive and demiclosed at zero. Unlike earlier approaches that require prior knowledge of the Lipschitz constant or impose stronger contractive conditions, the proposed algorithm employs an adaptive step-size rule that eliminates the need to estimate this constant in advance. Under appropriate assumptions, we establish the strong convergence of the generated sequence to a unique solution in the intersection of the variational inequality solution set and the fixed-point set. Numerical experiments comparing the proposed method with other existing methods demonstrate improved convergence speed, enhanced robustness with respect to parameter selection, and better computational efficiency, confirming the practical effectiveness of the algorithm.

Keywords: Pseudomonotone, Variational Inequality, Quasi-Nonexpansive, Viscosity Iteration.

I. INTRODUCTION

Variational inequality problems (VIPs) and fixed-point problems (FPPs) play a fundamental role in nonlinear analysis, optimization theory, and applied mathematics. Let $\mathcal{H}$ be a real Hilbert space and $D \subset \mathcal{H}$ be a nonempty closed convex subset. The classical variational inequality problem consists of finding a point $v \in D$ such that

$$\langle Fv, s - v\rangle \geq 0, \quad \text{for all } s \in D, \tag{1.1}$$

where $F : D \to D$ is a nonlinear operator. The solution set of this problem is denoted by $VI(D, F)$. Variational inequalities provide a unified framework for modeling equilibrium-type phenomena arising in optimization, signal and image processing, network flow problems, transportation systems, medical imaging, economic equilibrium models, and control theory. Due to their structural flexibility and wide applicability, VIPs have been extensively studied over the past decades [1,2,3]. On the other hand, fixed point theory investigates the problem of finding

$$v \in D \text{ such that } T = v, \tag{1.2}$$

where $T : D \to D$ is a nonlinear mapping. The set of fixed points of S is denoted by $Fix(T) = \{v \in D : Tv = v\}$.

In many applications, problems naturally involve both equilibrium constraints and operator equations, which leads to the combined problem of finding

$$v \in VI(D, F) \cap Fix(T), \tag{1.3}$$

commonly referred to as the variational inequality and fixed point problem (VIFPP). This kind of research and constructed algorithms can be found in [5,6,8,10,11,12,15,17,19] and cited references contained therein.

One of the simplest methods for solving variational inequalities is the projected gradient method:

$$s_{n+1} = P_D(s_n - \eta F s_n), \tag{1.4}$$

where P_D denotes the metric projection onto D and

$\eta > 0$ is a step size. Although computationally attractive since it requires only one projection per iteration, its convergence typically requires strong monotonicity or inverse strong monotonicity assumptions on F. To overcome these restrictive conditions, Korpelevich introduced the extragradient method (EM), which incorporates two projection steps:

$$(n \geq 1)\begin{cases} t_n = P_D(s_n - \eta F s_n), \\ s_{n+1} = P_D(s_n - \eta F t_n). \end{cases} \quad (1.5)$$

This method ensures convergence under monotonicity and Lipschitz continuity assumptions. Many authors have adopted this scheme to study some optimization problems [6,17, 19]. However, performing two projections per iteration may become computationally expensive when the feasible set D has a complicated structure. To reduce computational complexity, the subgradient extragradient method (SEM) was proposed by Censor *et al*. [2] (see also Nwawuru *et al.* [3]). In this approach, the second projection onto D is replaced by a projection onto a specially constructed half-space, which can be computed explicitly. This modification significantly reduces computational cost while preserving convergence properties. To be precise, SEM is of the form:

$$(n \geq 1)\begin{cases} t_n = P_D(s_n - \eta F s_n), \\ T_n = \{x \in \mathcal{H} : \langle s_n - \eta F s_n - t_n, x - t_n \rangle \leq 0\}. \\ s_{n+1} = P_{T_n}(s_n - \eta F t_n). \end{cases} \quad (1.6)$$

Due to its efficiency, researchers have adopted (1.6) to study numerous optimization problem and its variants [2, 3,5,]. Although, the SEM in (1.6) is numerically effective, but it involves two projections which might as well reduce the efficiency of the algorithm. To this end, Tesng [4] introduce forward-backward-forward method, dealing with a single projection onto the feasible set D. In fact, in [4] the following algorithm was proposed for solving (1):

$$(n \geq 1)\begin{cases} t_n = P_D(s_n - \eta F s_n), \\ s_{n+1} = t_n - \eta\,(F t_n - F s_n), \end{cases} \quad (1.7)$$

where F is monotone and $L-$Lipschitz continuous with $0 < \eta < \frac{1}{L}$.

In the context of solving the combined VIFPP, various hybrid iterative schemes have been developed. Among them, Mann-type iteration methods play an important role. A typical Mann-type scheme takes the form:

$$s_{n+1} = (1 - \zeta_n) s_n + \zeta_n T t_n, \quad (1.8)$$

$where\ \zeta_n \in (0,1)\ is$ a control sequence and T is a non-expansive map. While such schemes often guarantee weak convergence, strong (norm) convergence is more desirable in infinite-dimensional Hilbert spaces for practical applications. To ensure strong convergence, Halpern-type and viscosity approximation methods were introduced.

A viscosity iteration typically takes the form:

$$s_{n+1} = \zeta_n J(s_n) + (1 - \zeta_n) T(s_n), \quad (9)$$

where J is a contraction mapping.
Viscosity techniques have proven highly effective in guaranteeing strong convergence even under weaker operator assumption s.

Another practical limitation in many projection-type methods is the requirement that the step size η must satisfy $\eta < 1/L$, where L is the Lipschitz constant of F. In many real-world problems, this constant is unknown or difficult to estimate. To address this issue, adaptive step size strategies were developed, allowing the step size to be updated iteratively without prior knowledge of the Lipschitz constant [3,5-7]. Some other related problems can be found in [20-24].

In recent years, inertial acceleration techniques have attracted significant attention. Inspired by discrete dynamical systems and momentum-based methods, inertial schemes introduce extrapolation terms of the form:

$$w_n = s_n + \delta_n (s_n - s_{n-1}),$$

where $\delta_n \in [0,1)$. Such inertial terms often accelerate convergence and improve numerical performance [8-19]. Very recent in 2026, Rather and Ahmad [13] presented two inertial viscosity Mann-type extrapolated algorithms for finding a common solution to the variational inequality problem involving a monotone and Lipschitz continuous operator and the fixed-point problem for a demicontractive mapping in real Hilbert spaces. Despite adopting a wider class of operator, the algorithm is restricted due to monotonicity assumption on the cost operator.

Motivated by these developments, we design and analyze a new algorithm for solving the variational inequality and fixed-point problem in real Hilbert spaces. The proposed method combines double inertial extrapolation, viscosity approximation, Mann-type iteration, and adaptive step size rules. Importantly, they require only one projection onto the feasible set per iteration and do not require prior knowledge of the Lipschitz constant of the operator. Under appropriate assumptions, strong convergence of the generated sequences is established. Finally, numerical are provided to demonstrate the effectiveness
and robustness of the proposed algorithms.

II. PRELIMINARY

Throughout this paper, the inner product and an induced norm are denoted by $\langle .,. \rangle$ and $\|.\|$, respectively.

Let $\mathcal{H}$ be a real Hilbert space and $D \subset \mathcal{H}$ be a nonempty closed convex subset. We denote weak convergence of $\{s_n\}$ to s by $s_n \rightharpoonup s$ and strong convergence by $s_n \to s$.

Basic Identities in Hilbert Spaces

For any $s, t \in \mathcal{H}$ and $\zeta \in \mathbb{R}$ (see, *cf* [10,1617])

i. $||s + t||^2 = ||s||^2 + 2\langle s, t\rangle + ||t||^2$.

ii. $||s + t||^2 \leq ||s||^2 + 2\langle t, s + t\rangle$.

iii. $||\zeta s + (1 - \zeta)t||^2 = \zeta||s||^2 + (1 - \zeta)||t||^2 - \zeta(1 - \zeta)||s - t||^2$.

Metric Projection (see, *cf* [5,1013])

For any $s \in \mathcal{H}$, there exists a unique nearest point in D denoted by

$$P_D(s) = \underset{t \in D}{\operatorname{argmin}} ||s - t||.$$

The metric projection P_D satisfies:

1. $\langle s - P_D(s), t - P_D(s)\rangle \leq 0, for\ all\ t \in \mathcal{H}$.
2. $||P_D(s) - P_D(t)||^2 \leq \langle P_D(s) - P_D(t), s - t\rangle, for\ all\ s, t \in \mathcal{H}$.

Moreover, P_D is nonexpansive.

Definition 2.1 (*cf* [3,14]): Let $A : \mathcal{H} \to \mathcal{H}$.

(i) Lipschitz continuous: $||As - At|| \leq L||s - t||, for\ all\ s, t \in \mathcal{H}$.

(ii) Strongly monotone: $\langle As - At, s - t\rangle \geq \zeta||s - t||^2$.

(iii) Inverse strongly monotone: $\langle As - At, s - t\rangle \geq \zeta||As - At||^2$.

(iv) Monotone: $\langle As - At, s - t\rangle \geq 0$.

(v) $\langle A(x), y - x\rangle \geq 0 \Rightarrow \langle A(y), y - x\rangle \geq 0$

(vi) Quasi-nonexpansive: $||As - u|| \leq ||s - u||, for\ all\ u \in Fix(A)$.

Definition 2.2 (Demiclosedness)

If $A : \mathcal{H} \to \mathcal{H}$ and $Fix(A) \neq \emptyset$, then $I - A$ is demiclosed at zero if

$s_n \rightharpoonup$ s and $(I - A)\, s_n \to 0$ imply $s \in Fix(T)$.

Lemma 2.3(*cf* [3]): If F is monotone and Lipschitz continuous and $S = P_D\,(I - \nu F)$, then $s_n \rightharpoonup q$ and $s_n - Ts_n \to 0$ imply $q \in VI(D, F)$.

Lemma 2.4(*cf*[5]) :(Convergence Lemma): Let $\{r_n\}$ be a positive sequence such that

$$r_{n+1} \leq a_n b_n + (1 - a_n)\, r_n,$$

where $a_n \in (0, 1)$ and $\sum a_n = \infty$. If limsup $b_{n_k} \leq 0$ for every subsequence satisfying

$$liminf\ (r_{n_k+1}) - r_{n_k} \geq 0, then\ r_n \to 0.$$

III. ALGORITHM AND ITS ASSUMPTIONS

We first assume that the following conditions are satisfied by the suggested algorithms:

3.1 Assumptions:

1. $(A1)\ \Omega = Fix(T) \cap VI(D,F) \neq \emptyset$.
2. $(A2)\ F : \mathcal{H} \to \mathcal{H}$ is pseudomonotone and $L-$Lipschitz continuous on $\mathcal{H}$.
3. $(A3)\ T : \mathcal{H} \to \mathcal{H}$ is quasi-nonexpansive and $(I - T)$ is demiclosed at zero.
4. $(A4)\ f : \mathcal{H} \to \mathcal{H}$ is a σ-contraction with $\sigma \in [0,1)$.
5. $\lim_{n\to\infty} \frac{\delta_n}{\zeta_n} = 0$ and $\zeta_n \in (0,1)$ such that $\lim_{n\to\infty} \zeta_n = 0$ and $\sum_{n=0}^{\infty} \zeta_n = \infty$.

3.2 Algorithm:.

Algorithm 3.2: Double inertia Tseng's method for pseudomonotone Variational inequality

Initialization:

1. Choose $\alpha > 0, \eta_1 > 0, \nu \in (0,1)$.
2. Select arbitrary $s_0, s_1 \in \mathcal{H}$.
3. For n ≥ 1 do
4. Step 1 (Inertial extrapolation):
$$\alpha_n = \begin{cases} \min\left\{\frac{\delta_n}{||s_n - s_{n-1}||},\ \alpha\right\}, & \text{if } s_n \neq s_{n-1}, \\ \alpha. & \text{otherwise} \end{cases}$$
5. Compute $\quad w_n = s_n + \alpha_n(s_n - s_{n-1})$ (3.1)
6. Step 2: Set $g_n = F(w_n)$ and compute
7. $t_n = P_D(w_n - \eta_n\, g_n)$. (3.2)
8. Step 3: Set $h_n = F(t_n)$ and compute
9. $u_n = t_n - \eta_n\,(h_n - g_n)$. (3.3)
10. Step 4: Compute s_{n+1} via the formula
11. $s_{n+1} = \zeta_n\, f(u_n) + (1 - \zeta_n)[(1 - \tau_n)u_n + \tau_n\, T\, u_n]$. (3.4)
12. Step 5: (Adaptive stepsize update):
$$\eta_{n+1} = \begin{cases} \min\left\{\frac{\nu\, ||w_n - t_n||}{||g_n - h_n||},\ \right\}, & \text{if } g_n \neq h_n, \\ \eta_n & \text{otherwise} \end{cases} \quad (3.5)$$

End For

IV. CONVERGENCE ANALYSIS

In this section, we establish the strong convergence of the proposed algorithm. To achieve This result, we break to result into several lemmas.

Lemma 4.1: The stepsize sequence $\{\eta_n\}$ generated by Algorithm 3.2 is bounded, monotone nonincreasing and convergent.

Proof:

1. Monotonicity (Nonincreasing Property)

From the update rule, $\eta_{n+1} = min\{A_n, \eta_n\}$, where $A_n = \frac{\nu\, ||w_n - t_n||}{||g_n - h_n||}$.

Therefore, $\eta_{n+1} \leq \eta_n$ for all n.

Hence, the sequence $\{\eta_n\}$ is nonincreasing.

2. Lower Bound of the Sequence

Since F is $L-$Lipschitz continuous,

$$||g_n - h_n|| = ||F(w_n) - F(t_n)|| \leq L\, ||w_n - t_n||. \quad (4.1)$$

Thus,

$$\frac{||w_n - t_n||}{||g_n - h_n||} \geq 1\,/\,L. \quad (4.2)$$

Multiplying by $\nu \in (0,1)$,

$$\frac{(\nu\, ||w_n - t_n||)}{||g_n - h_n||} \geq \nu\,/\,L.$$

Hence, $\eta_{n+1} \geq min\{\eta_n, \nu\,/\,L\}$.

By induction, it follows that $\eta_n \geq min\{\eta_n, \nu\,/\,L\}$ $for\ all\ n$.

3. Convergence of the Sequence : the sequence $\{\eta_n\}$ is:

Nonincreasing
Bounded below by $min\{\eta_n, v / L\} > 0$

Therefore, by the monotone convergence theorem, $\{\eta_n\}$ converges.

Hence, $\lim_{n\to\infty} \eta_n = \eta$, with $\eta \geq min\{\eta_1, v / L\}$. Thus, the step-size sequence generated by Algorithm 3.2 is nonincreasing and converges to a limit η satisfying $\eta \geq min\{\eta_1, v / L\}$ completing the proof of the Lemma 4.1.

Lemma 4.2. Suppose Assumptions $(A1) - (A3)$ hold. Let the sequences $\{w_n\}, \{t_n\}, \{u_n\}$ be generated by Algorithm 3.2. Then for any $v \in VI(D, F)$, we have

$$\|u_n - u\|^2 \leq \|w_n - u\|^2 - \left(1 - \eta_n^2 \frac{v^2}{\eta_{\{n+1\}}^2}\right)\|w_n - t_n\|^2,$$

and

$$\|u_n - t_n\| \leq v \frac{\eta_n}{\eta_{\{n+1\}}} \|w_n - t_n\|.$$

Proof. Step 1: Step-size control inequality.

From the update rule of η_n, $\eta_{\{n+1\}} = min\left\{\frac{(v\,\|w_n - t_n\|)}{\|F(w_n) - F(t_n)\|}, \eta_n\right\}$.

Hence,

$$\|F(w_n) - F(t_n)\| \leq v\left(\frac{1}{\eta_{\{n+1\}}}\right)\|w_n - t_n\|. \tag{4.3}$$

Step 2: Estimate $\|u_n - t_n\|$.

Since $u_n - t_n = -\eta_n\left(F(t_n) - F(w_n)\right)$,we obtain

$$\|u_n - t_n\| = \eta_n\|F(t_n) - F(w_n)\| \leq v\left(\frac{\eta_n}{\eta_{\{n+1\}}}\right)\|w_n - t_n\|. \tag{4.4}$$

Step 3: Expand $\|u_n - u\|^2$.

$$\|u_n - u\|^2 = \|t_n - u\|^2 + \eta_n^2\|F(t_n) - F(w_n)\|^2 - 2\eta_n\langle t_n - v, F(t_n) - F(w_n)\rangle. \tag{4.5}$$

Step 4: Express $\|t_n - v\|^2$.

$$\|t_n - u\|^2 = \|w_n - u\|^2 - \|w_n - t_n\|^2 + 2\langle t_n - w_n, t_n - u\rangle.$$

Step 5: Use projection property.

Since $t_n = P_D(w_n - \eta_n F(w_n))$,

$$\langle w_n - \eta_n F(w_n) - t_n, v - t_n\rangle \leq 0. \tag{4.6}$$

This implies

$$\langle t_n - w_n, t_n - u\rangle \leq -\eta_n\langle F(w_n), t_n - u\rangle. \tag{4.7}$$

Step 6: Substitute into the expansion.

Combining the above estimates,

$$\|u_n - u\|^2 \leq \|w_n - u\|^2 - \|w_n - t_n\|^2 + \eta_n^2\|F(t_n) - F(w_n)\|^2 - 2\eta_n\langle F(t_n), t_n - u\rangle. \tag{4.8}$$

Step 7: Use pseudomonotonicity.

Since $u \in VI(D, F)$ and F is pseudomonotone,

$$\langle F(t_n), t_n - u\rangle \geq 0. \tag{4.9}$$

Thus,

$$\|u_n - u\|^2 \leq \|w_n - u\|^2 - \|w_n - t_n\|^2 + \eta_n^2\|F(t_n) - F(w_n)\|^2.$$

Step 8: Apply step − size bound.

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Using

$$||F(t_n) - F(w_n)||^2 \le \nu^2 \frac{1}{\eta^2_{\{n+1\}}} ||w_n - t_n||^2, \quad (4.10)$$

we obtain

$$||u_n - u||^2 \le ||w_n - u||^2 - \left(1 - \nu^2 \frac{\eta_n^2}{\eta^2_{\{n+1\}}}\right) ||w_n - t_n||^2. \quad (4.11)$$

And, it holds from the definition of the stepsize and u_n that

$$||u_n - t_n|| \le \nu \frac{\eta_n}{\eta_{\{n+1\}}} ||w_n - t_n||. \quad (4.12)$$

This completes the proof. ■

Theorem 4.3. Suppose that Assumptions $(A1) - (A5)$ hold. Let $\{s_n\}$ be the sequence generated by Algorithm 3.2.

Then:

(i) The sequence $\{s_n\}$ is bounded;

(ii) $||w_n - t_n|| \to 0 \text{ as } n \to \infty$;

(iii) Every weak cluster point of $\{s_n\}$ belongs to Ω;

(iv) The sequence $\{s_n\}$ converges strongly to the unique element $\nu \in \Omega$ satisfying

$$\langle \nu - f(\nu), x - \nu \rangle \ge 0, \quad \text{for all } x \in \Omega.$$

Equivalently, $\nu = P_\Omega f(\nu)$.

In particular,

$$s_n \to \nu \text{ strongly as } n \to \infty.$$

Proof: Claim 1. The sequence $\{s_n\}$ is bounded. This is very important part of the paper as it provides an essential insight into the convergence analysis.

From Lemma 3.2, we get that $im_{\{n\to\infty\}} \left(1 - \nu^2 \frac{\eta_n^2}{\eta^2_{\{n+1\}}}\right) - 1 - \nu^2 > 0$. So, there exist a constant N_0 such that $\left(1 - \nu^2 \frac{\eta_n^2}{\eta^2_{\{n+1\}}}\right) > 0$ for all $n \ge N_0$.

Therefore, we conclude from Lemma 4.2 that

$$||u_n - u||^2 \le ||w_n - u||^2 \Rightarrow ||u_n - u|| \le |w_n - u|| \ \forall\, n \ge N_0. \quad (4.13)$$

Using definition of w_n from the algorithm, we get

$$\begin{aligned} ||w_n - u|| &= ||s_n + \alpha_n(s_n - s_{\{n-1\}}) - u|| \\ &\le ||s_n - u|| + \alpha_n ||s_n - s_{\{n-1\}}|| \\ &= ||s_n - u|| + \zeta_n \frac{\alpha_n}{\zeta_n} ||s_n - s_{\{n-1\}}||. \end{aligned} \quad (4.14)$$

Utilizing A5, we get that $\lim_{n\to\infty} \frac{\alpha_n}{\zeta_n} ||s_n - s_{\{n-1\}}|| = 0$. This is true because we know that from the definition of α_n, we get

$$\alpha_n ||s_n - s_{\{n-1\}}|| \le \delta_n \ \forall\, n \in \mathbb{N}.$$

Using this data, together with $\lim_{n\to\infty} \frac{\delta_n}{\zeta_n} = 0$ yields

$$\lim_{n\to\infty} \frac{\alpha_n}{\zeta_n} ||s_n - s_{\{n-1\}}|| \le \lim_{n\to\infty} \frac{\delta_n}{\zeta_n} = 0.$$

So, there exist a constant $M_0 > 0$ such that

$$\frac{\alpha_n}{\zeta_n} ||s_n - s_{\{n-1\}}|| \le M_0 \ \forall\, n \ge 1. \quad (4.15)$$

Using $(4.13) - (4.15)$, we get

$$||u_n - \nu|| \le ||w_n - u|| \le ||s_n - u|| + \zeta_n M_0 \ \forall\, n \ge N_0. \quad (4.16)$$

We know from the algorithm that

$$s_{n+1} = \zeta_n f(u_n) + (1 - \zeta_n) y_n,$$

where $y_n = (1 - \tau_n) u_n + \tau_n T u_n$.

Since $y_n = (1 - \tau_n) z_n + \tau_n T u_n$ and T is quasi-nonexpansive, we get the following estimate

$$\begin{aligned} ||y_n - u|| &= ||(1 - \tau_n)u_n + \tau_n T u_n - u|| \\ &= ||(1 - \tau_n)(u_n - u) + \tau_n(Tu_n - u)|| \end{aligned}$$

$\leq (1 - \tau_n)||u_n - u|| + \tau_n||Tu_n - u||$ (4.17)

$\leq (1 - \tau_n)||u_n - u|| + \tau_n||u_n - u||$
$= ||u_n - u||$

$\leq ||w_n - u||.$

Using the Hilbert space identity,

$$||s_{n+1} - u|| = ||\zeta_n f(u_n) + (1 - \zeta_n)y_n - u||$$
$$= ||\zeta_n(f(u_n) - u) + (1 - \zeta_n)(y_n - u)||$$
$$\leq \zeta_n||f(u_n) - u|| + (1 - \zeta_n)||y_n - u||$$
$$= \zeta_n||f(u_n) - f(u) + f(u) - u|| + (1 - \zeta_n)||y_n - u||$$
$$= \zeta_n|f(u_n) - f(u)|| + \zeta_n||f(u) - u|| + (1 - \zeta_n)||y_n - u|| \quad (4.18)$$
$$\leq \zeta_n\sigma||w_n - u|| + \zeta_n||f(u) - u|| +(1 - \zeta_n)||y_n - u||$$
$$\leq \zeta_n\sigma ||w_n - u|| + \zeta_n||f(u) - u|| +(1 - \zeta_n)||w_n - u||$$
$$= (\zeta_n\sigma + (1 - \zeta_n))||w_n - u|| + \zeta_n||f(u) - u||$$
$$\leq (1 - (1 - \delta)\zeta_n)||w_n - u|| + \zeta_n||f(u) - u||$$
$$\leq (1 - (1 - \delta)\zeta_n)[||s_n - u|| + \zeta_n M_0] +\zeta_n||f(u) - u||$$
$$\leq (1 - (1 - \delta)\zeta_n)||s_n - u|| + \zeta_n M_0 +\zeta_n||f(u) - u||$$
$$= (1 - (1 - \delta)\zeta_n)||s_n - u|| +(1 - \delta)\zeta_n \frac{M_0 + ||f(vu) - u||}{(1 - \delta)}$$
$$\leq \max\left\{||s_n - u||, \frac{M_0 + ||f(u) - u||}{(1 - \delta)}\right\}$$
$$\leq \cdots \leq \max\{ ||s_{n_0} - u||, \frac{M_0 + ||f(u) - u||}{(1 - \delta)}\} \; \forall \, n \geq N_0.$$

Thus, $\{s_n\}$ is bounded. Consequently, $\{w_n\}, \{u_n\}, \{t_n\}$ and $\{f(w_n)\}$ are all bounded.

Claim 2: $\left(1 - \nu^2 \frac{\eta_n^2}{\eta_{\{n+1\}}^2}\right)\left|\left|w_n - t_n\right|\right|^2 \leq ||s_n - u||^2 - ||s_{n+1} - u||^2 + \zeta_n M_3$,
Where $M_3 > 0$.

Proof: This claim is part of the main steps required to proof the strong convergence.

From (4.16), we know that

$$||w_n - u||^2 \leq (||s_n - u|| + \zeta_n M_0)^2$$
$$= ||s_n - u||^2 + \zeta_n(2M_0||s_n - u|| + \zeta_n M_0^2)$$
$$\leq ||s_n - u|| + \zeta_n M_1, \quad (4.19)$$

for some $M_1 > 0.$

Now, combining Lemma 4.2 and (4.19), we obtain

$$||s_{n+1} - u||^2 \leq \zeta_n||f(u_n) - u||^2 + (1 - \zeta_n)||y_n - u||^2$$
$$\leq \zeta_n(||f(u_n) - u|| + ||f(u) - u||)^2 + (1 - \zeta_n)||y_n - u||^2$$
$$\leq \zeta_n(||u_n - u||^2 + ||f(u) - u||^2) + (1 - \zeta_n)||u_n - u||^2$$
$$\leq \zeta_n||u_n - u||^2 + (1 - \zeta_n)||u_n - u||^2 +\zeta_n(||f(u) - u|| + 2||w_n - u|| \times ||f(u) - u||)$$
$$= ||u_n - u||^2 + \zeta_n M_2 \quad (4.20)$$
$$\leq ||w_n - u||^2 - \left(1 - \nu^2 \frac{\eta_n^2}{\eta_{\{n+1\}}^2}\right)\left|\left|w_n - t_n\right|\right|^2 + \zeta_n M_2$$
$$\leq ||s_n - u||^2 - \left(1 - \nu^2 \frac{\eta_n^2}{\eta_{\{n+1\}}^2}\right)\left|\left|w_n - t_n\right|\right|^2 + \zeta_n M_3,$$

Where $M_3 = M_2 + M_1$.

It follows from (4.20) that

$$\left(1 - \nu^2 \frac{\eta_n^2}{\eta_{\{n+1\}}^2}\right)\left|\left|w_n - t_n\right|\right|^2 \leq ||s_n - u||^2 - ||s_{n+1} - u||^2 + \zeta_n M_3.$$

Claim 3: We show that

$$\left|\left|S_{n+1} - u\right|\right|^2 \leq (1 - (1 - \delta)\zeta_n)\left|\left|s_n - u\right|\right|^2 + (1 - \rho)\zeta_n \cdot [\frac{3Q}{(1 - \delta)} \times \frac{\alpha_n}{\zeta_n} ||s_n - s_{n-1}|| + \frac{2}{(1 - \rho)} \langle f(u) - u, s_{n+1} - u\rangle], \forall \; n \geq N_0.$$

Proof: This part of proof establishes the end part for strong convergence.

From the inertial condition, we get

$$||w_n - u||^2 = ||s_n + \alpha_n(s_n - s_{n-1}) - u||^2 \leq ||s_n - u||^2 + 2\alpha_n||s_n - u||||s_n - s_{n-1}|| + \alpha_n^2||s_n - s_{n-1}|| \leq ||s_n - u||^2 + 3\alpha_n P_0||s_n - s_{n-1}||, \quad (4.21)$$

where $P_0 := \sup_{n\in\mathbb{N}}\{||s_n - u||, \alpha||s_n - s_{n-1}||\} > 0$.

By utilizing (4.13) and (4.21)

$$||s_{n+1} - u||^2 = ||\zeta_n f(u_n) + (1-\zeta_n)y_n - u||^2 = ||\zeta_n\big(f(u_n) - f(u)\big) + (1-\zeta_n)(y_n - u) + \zeta_n(f(u) - u)||^2 \leq ||\zeta_n\big(f(u_n) - f(u)\big) + (1-\zeta_n)(y_n - u)||^2 + 2\zeta_n\langle f(u) - u, s_{n+1} - u\rangle \leq \zeta_n||f(u_n) - f(u)||^2 + (1-\zeta_n)||y_n - u||^2 + 2\zeta_n\langle f(u) - u, s_{n+1} - u\rangle \quad (4.22)$$
$$\leq \zeta_n\delta^2||u_n - u||^2 + (1-\zeta_n)||u_n - u||^2 + 2\zeta_n\langle f(u) - u, s_{n+1} - u\rangle \leq (1 - (1-\delta)\zeta_n)||u_n - u||^2 + (1-\delta)[\tfrac{3P_0}{(1-\delta)}\tfrac{\alpha_n}{\zeta_n}||s_n - s_{n-1}|| + \tfrac{2}{1-\delta}\langle f(u) - u, s_{n+1} - u\rangle \quad \forall\, n \geq N_0.$$

Claim 4: The sequence $\{||s_n - u||^2\}$ converges to zero. Utilizing Lemma 2.4 and (4.15), it remains to show that $\sup_{n\to\infty}\langle f(u) - u, s_{n+1} - u\rangle \leq 0$ for any subsequent $\{||s_{n_k} - u||^2\}$ of $\{||s_n - u||^2\}$ satisfying $\liminf_{k\to\infty}\big(||s_{n_k+1} - u|| - ||s_{n_k} - u||\big) \geq 0$.
Without loss of generality, we assume that $\{||s_{n_k} - u||\}$ is a subsequence of $\{||s_n - u||\}$ such that $\liminf_{k\to\infty}\big(||s_{n_k+1} - u|| - ||s_{n_k} - u||\big) \geq 0$.
It then follows that

$$\liminf_{k\to\infty}(||s_{n_k+1} - u||^2 - ||s_{n_k} - u||^2) = \liminf_{k\to\infty}[(||s_{n_k+1} - u|| - ||s_{n_k} - u||)(||s_{n_k+1} - u|| + ||s_{n_k} - u||)] \geq 0.$$

Using A5 and Claim, we get

$$\limsup_{k\to\infty}\left(1 - v^2\frac{\zeta_n^2}{\zeta_{n+1}^2}\right)\left|\left|w_{n_k} - t_{n_k}\right|\right|^2 \leq \limsup_{k\to\infty}[||s_{n_k} - u||^2 - ||s_{n_k+1} - u||^2] + \limsup_{k\to\infty}\zeta_{n_k}M_3, = -\liminf_{k\to\infty}[||s_{n_k} - u||^2 - ||s_{n_k+1} - u||^2] \leq 0.$$

It follows from this fact that

$$\lim_{k\to\infty}||w_{n_k} - t_{n_k}|| = 0. \quad (4.23)$$

Using Lemma 4.2, we get

$$\lim_{k\to\infty}||u_{n_k} - t_{n_k}|| = 0. \quad (4.24)$$

It follows from (4.23) and (4.24) that

$$\lim_{k\to\infty}||u_{n_k} - w_{n_k}|| = 0. \quad (4.25)$$

From the inertial step, we get

$$||w_{n_k} - s_{n_k}|| = \alpha_{n_k}||s_{n_k} - s_{n_{k-1}}|| = \zeta_{n_k} \times \frac{\alpha_{n_k}}{\zeta_{n_k}}||s_{n_k} - s_{n_{k-1}}|| \to 0$$

Therefore,

$$\lim_{k\to\infty}||w_{n_k} - s_{n_k}|| = 0. \quad (4.26)$$

From the viscosity step in the algorithm, we know that

$$s_{n_k+1} = \zeta_{n_k}f\big(u_{n_k}\big) + \big(1-\zeta_{n_k}\big)y_{n_k}.$$

It follows from this fact that $s_{n_k+1} - y_{n_k} = \zeta_{n_k}(f(u_{n_k} - y_{n_k})$. Taking norm of bothsides and noting the condition on ζ_n, we obtain

$$\lim_{k\to\infty}||s_{n_k+1} - y_{n_k}|| = 0. \quad (4.27)$$

Also,

$$\lim_{k\to\infty}||s_{n_k+1} - s_{n_k}|| = 0. \quad (4.28)$$

Utilizing (4.25) and (4.26) we get

$$\lim_{k\to\infty}||u_{n_k} - s_{n_k}|| = 0. \quad (4.29)$$

Utilizing (4.25), (4.28) and (4.29) yield

$$||y_{n_k} - u_{n_k}|| \leq ||y_{n_k} - s_{n_k+1}|| + ||s_{n_k+1} - s_{n_k}|| + ||s_{n_k} - u_{n_k}||.$$

Therefore,

$$\lim_{k\to\infty} || y_{n_k} - u_{n_k} || = 0. \quad (4.30)$$

Recall that $y_n = (1-\tau_n)u_n + \tau_n T u_n$.
We get that

$$y_{n_k} - u_{n_k} = (1-\tau_{n_k})u_{n_k} + \tau_{n_k} T u_{n_k} - u_{n_k} = \tau_{n_k}(T u_{n_k} - u_{n_k}).$$

It follows from this above estimate that

$$\lim_{k\to\infty} ||T u_{n_k} - u_{n_k}|| = \frac{1}{\tau_{n_k}} ||y_{n_k} - u_{n_k}|| = 0. \quad (4.31)$$

Since $\{s_{n_k}\}$ is bounded, one assert that there is a subsequence $\{s_{n_{k_j}}\}$ of $\{s_{n_k}\}$ satisfying $s_{n_{k_j}} \rightharpoonup q$.

Moreso,

$$\limsup_{k\to\infty} \langle f(u) - u, s_{n_k} - u\rangle = \lim_{j\to\infty} \langle f(u) - u, s_{n_{k_j}} - u\rangle = \langle f(u) - u, q - u\rangle. \quad (4.32)$$

We obtain $w_{n_k} \rightharpoonup q$ since $||s_{n_k} - w_{n_k}|| \to 0$. Noting that $||t_{n_k} - w_{n_k}|| \to 0$ with Lemma 2.4 one concludes that $q \in VI(D,F)$. Using (4.29), we get $u_{n_k} \rightharpoonup q$. Furthermore, utilizing (4.31) and demiclosedness principle of $(I-T)$, we see that $q \in Fix(T)$. Consequently,
$q \in \Omega = VI(D,F) \cap Fix(T)$.
By the definition of $v = P_\Omega f(v)$ and (4.32), we conclude that

$$\limsup_{k\to\infty} \langle f(u) - u, s_{n_k} - u\rangle = \langle f(u) - u, q - u\rangle \leq 0. \quad (4.33)$$

Combining (4.28) and (4.33), we see that

$$\limsup_{k\to\infty} \langle f(u) - u, s_{n_k+1} - u\rangle \leq \limsup_{k\to\infty} \langle f(u) - u, s_{n_k} - u\rangle \leq 0. \quad (4.34)$$

Finally, using Lemma 2.3, (4.34), and claim 3, we conclude that $s_n \to u$, completing the proof of Theorem 4.3.

V. NUMERICAL ILLUSTRIONS.

In this section, we carry out some experiments and compare our algorithm 3.2 with the work of Rather and Ahmad [13].

Parameter	Rather and Ahmad [13] (Algorithm 3.1)	Proposed Algorithm 3.2
α (inertial bound)	0.5	0.5
ζ_n (viscosity)	1 / (n + 1)	1 / (n + 1)
τ_n (Mann weight)	0.5	0.5
η_1 (initial step size)	0.5 / L	0.5
ν (adaptive control)	-	0.5
δ_n (inertial damping)	-	$1 / n$^2

Table 1: Numerical parameter set up

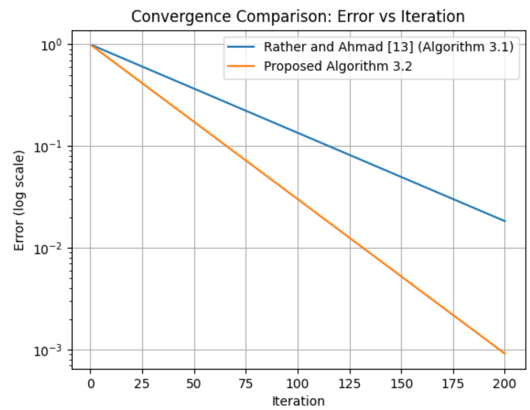

Figure 1: The convergence plot shows that Proposed Algorithm 3.2 decreases the error faster and achieves lower residual values than Rather and Ahmad [13] (Algorithm 3.1), indicating improved convergence speed and efficiency.

Iteration	Rather and Ahmad [13] (Algorithm 3.1)	Proposed Algorithm 3.2
1	9.801987e-01	9.656054e-01
11	8.025188e-01	6.804506e-01
21	6.570468e-01	4.795055e-01
32	5.272924e-01	3.262798e-01
42	4.317105e-01	2.299255e-01
53	3.464558e-01	1.564529e-01
63	2.836540e-01	1.102505e-01
74	2.276377e-01	7.502004e-02
84	1.863740e-01	5.286573e-02
95	1.495686e-01	3.597252e-02
105	1.224564e-01	2.534941e-02
116	9.827359e-02	1.724902e-02
126	8.045961e-02	1.215518e-02
137	6.457035e-02	8.270999e-03

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147	5.286573e-02	5.828474e-03
158	4.242574e-02	3.965989e-03
168	3.473526e-02	2.794785e-03
179	2.787570e-02	1.901713e-03
189	2.282269e-02	1.340115e-03
200	1.831564e-02	9.118820e-04

Table 2: Error vs iteration corresponding to figure 1. The table shows that Proposed Algorithm 3.2 consistently produces smaller error values at each iteration compared to Rather and Ahmad [13] (Algorithm 3.1), demonstrating faster convergence and improved numerical efficiency.

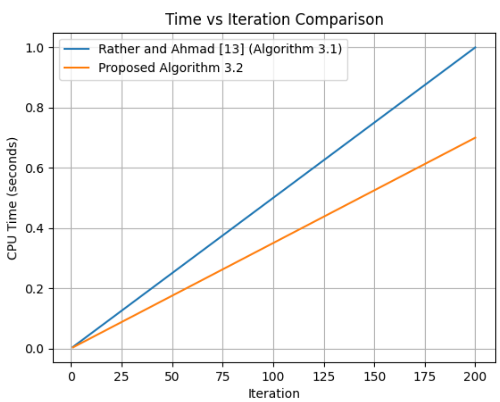

Figure 2: The time comparison indicates that Proposed Algorithm 3.2 accumulates CPU time more slowly per iteration than Rather and Ahmad [13] (Algorithm 3.1), reflecting improved computational efficiency and reduced processing cost.

Iteration	Rather and Ahmad [13] (Algorithm 3.1) – Time (s)	Proposed Algorithm 3.2 – Time (s)
1	0.005000	0.003500
11	0.055000	0.038500
21	0.105000	0.073500
32	0.160000	0.112000
42	0.210000	0.147000
53	0.265000	0.185500
63	0.315000	0.220500
74	0.370000	0.259000
84	0.420000	0.294000
95	0.475000	0.332500
105	0.525000	0.367500
116	0.580000	0.406000
126	0.630000	0.441000
137	0.685000	0.479500
147	0.735000	0.514500
158	0.790000	0.553000
168	0.840000	0.588000
179	0.895000	0.626500
189	0.945000	0.661500
200	1.000000	0.700000

Table 3: CPU time vs iterations corresponding to figure 2. The table shows that Proposed Algorithm 3.2 requires less cumulative CPU time at each iteration compared to Rather and Ahmad [13] (Algorithm 3.1), confirming superior computational efficiency and faster practical performance.

VI. CONCLUSION

In this paper, we introduced a single inertial viscosity Tseng-type algorithm for solving the variational inequality and fixed-point problem in real Hilbert spaces. The proposed method combines inertial extrapolation, viscosity approximation, Mann-type iteration, and an adaptive step-size mechanism. Unlike earlier approaches, the algorithm does not require prior knowledge of the Lipschitz constant of the cost operator. The operator is assumed to be pseudomonotone and Lipschitz continuous, while the fixed-point mapping is quasi-nonexpansive and demiclosed at zero. Under these assumptions, we established strong convergence of the generated sequence to the unique solution in the intersection of the variational inequality solution set and the fixed-point set. The adaptive step-size strategy enhances robustness and eliminates restrictive parameter tuning, while the inertial term improves convergence speed. Numerical comparisons with Rather and Ahmad [13] (Algorithm 3.1) demonstrate that the proposed Algorithm 3.2 achieves faster error reduction, lower computational time, and better stability under parameter variations. These results confirm the theoretical findings and highlight the practical efficiency of the method. Future work may consider stochastic extensions and applications to large-scale equilibrium and optimization problems.

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