© MAR 2026 | IRE Journals | Volume 9 Issue 9 | ISSN: 2456-8880
DOI: https://doi.org/10.64388/IREV9I9-1715023

On Fuzzy Soft Y-Class Operators in Fuzzy Soft Hilbert Spaces

A.M. NYONGESA
Department of Mathematics, Kibabii University, Kenya

Abstract- This paper introduces and studies the class of fuzzy soft Y-operators in fuzzy soft Hilbert spaces, extending the classical Y-class operators introduced by Uchiyama and Yoshino (1997) to the fuzzy soft setting. Building upon the foundation of fuzzy soft normal operators established by Dawood and Jabur (2021), we define fuzzy soft f_χ-operators through the condition $\uparrow f^\chi - \lambda f^\chi + \rho^\chi \leq K_\alpha^2(f^\chi - \hat{z})(f - \hat{z})$; for all $z^\chi \in C(A)$. We investigate fundamental properties including algebraic structure, spectral characteristics, and relationships with existing operator classes such as fuzzy soft normal, fuzzy soft self-adjoint, and fuzzy soft unitary operators. Several important theorems are proved, including a fuzzy soft Putnam-Fuglede type theorem for Y-class operators and compactness criteria. The results generalize and extend both the classical Y-operator theory and the fuzzy soft normal operator theory, providing a comprehensive framework for studying parameterized operator classes in fuzzy soft Hilbert spaces.

Keywords: Fuzzy Soft Hilbert Spaces, Y-Class Operators, Fuzzy Soft Normal Operators, Putnam-Fuglede Theorem, Spectral Theory.

I. INTRODUCTION

The theory of fuzzy sets, introduced by Zadeh (1965), and soft sets, developed by Molodtsov (1999), have revolutionized mathematical modeling under uncertainty. The fusion of these concepts into fuzzy soft sets by Maji et al. (2001) opened new avenues for handling imprecise and parameterized data structures. Subsequent developments include fuzzy soft topological spaces (Neog et al., 2012), fuzzy soft normed spaces (Beaula and Priyanga, 2015), and fuzzy soft Hilbert spaces (Faried et al., 2020a).

In operator theory, Uchiyama and Yoshino (1997) introduced the class Y of operators satisfying

$$| T^*T - TT^* | \leq K_\alpha^2 \, | T - \tilde{z} | \, (T - \tilde{z}),$$

for all $z \in C$, generalizing M-hyponormal and dominant operators. This class exhibits interesting properties including generalized Putnam-Fuglede theorems and compactness criteria.

Recently, Dawood and Jabur (2021) extended operator theory to fuzzy soft Hilbert spaces, introducing fuzzy soft normal operators $\hat{f}^*$satisfying $\hat{f}^*\hat{f} = \hat{f}\hat{f}^*$. Their work established fundamental properties and relationships with other fuzzy soft operator classes.

The present paper bridges these two research streams by introducing fuzzy soft Y-class operators which generalize both the classical Y-class operators and fuzzy soft normal operators. Our contributions include:
(i) Definition of fuzzy soft Y-operators in fuzzy soft Hilbert spaces.
(ii) Investigation of algebraic properties and operator calculus.
(iii) Proof of a fuzzy soft Putnam-Fuglede type theorem for Y-class operators.
(iv) Spectral analysis and compactness criteria.
(v) Relationships with existing fuzzy soft operator classes.
(vi) Applications to operator equations in fuzzy soft Hilbert spaces.

This work significantly extends the theory of Dawood and Jabur (2021) while importing powerful techniques from Uchiyama and Yoshino (1997) into the fuzzy soft setting.

II. PRELIMINARIES AND NOTATION

We recall essential definitions from fuzzy soft analysis. Let Xbe a universe set, Aa parameter set, and $P(X)$the power set of X.

Definition 2.1 (Zadeh, 1965). A fuzzy set $A^\sim$over $X$is characterized by a membership function

$$\mu_{A^\sim}: X \to [0,1].$$

© MAR 2026 | IRE Journals | Volume 9 Issue 9 | ISSN: 2456-8880
DOI: https://doi.org/10.64388/IREV9I9-1715023

Definition 2.2 (Molodtsov, 1999). A soft set over $X$with respect to $A$is a pair $(G, A)$where

$$G: A \to P(X).$$

Definition 2.3 (Maji et al., 2001). A fuzzy soft set over $X$is a soft set $(G, A)$where

$$G: A \to T_X$$

with

$$T = [0,1].$$

Definition 2.4 (Beaula and Priyanga, 2015). A fuzzy soft normed space $(X^{\sim}, \|\cdot\|)$is a fuzzy soft vector space with a mapping

$$\|\cdot\|: X^{\sim} \to R(A)$$

satisfying:

(i)

$$\| x^{\sim} \| \geq \tilde{0}$$

for all $x^{\sim} \in X^{\sim}$, with equality iff

$$x^{\sim} = \vartheta^{\sim}.$$

(ii)

$$\| r^{\sim} \cdot x^{\sim} \| = | r^{\sim} | \| x^{\sim} \|$$

for all $x^{\sim} \in X^{\sim}, r^{\sim} \in C(A)$.

(iii)

$$\| x^{\sim} + y^{\sim} \| \leq \| x^{\sim} \| + \| y^{\sim} \|$$

for all $x^{\sim}, y^{\sim} \in X^{\sim}$.

Definition 2.5 (Faried et al., 2020a). A fuzzy soft Hilbert space $(\widetilde{H}, (\cdot,\cdot))$is a complete fuzzy soft inner product space with inner product

$$(\cdot,\cdot): \widetilde{H} \times \widetilde{H} \to C(A).$$

Definition 2.6 (Faried et al., 2020b). A fuzzy soft linear operator

$$T^{\wedge}: \widetilde{H} \to \widetilde{H}$$

satisfies

$$T^{\wedge}(\tilde{\alpha}\, x^{\sim} + \tilde{\beta}\, y^{\sim}) = \tilde{\alpha}\, T^{\wedge}(x^{\sim}) + \tilde{\beta}\, T^{\wedge}(y^{\sim})$$

for all $x^{\sim}, y^{\sim} \in \widetilde{H}$and $\tilde{\alpha}, \tilde{\beta} \in C(A)$.

Definition 2.7 (Dawood and Jabur, 2021). A fuzzy soft normal operator $T^{\wedge}$satisfies

$$T^{\wedge}T^{\wedge *} = T^{\wedge *}T^{\wedge}.$$

We denote by

$$B(\widetilde{H})$$

the set of all bounded fuzzy soft linear operators on $\widetilde{H}$.

III. FUZZY SOFT Y_A-CLASS OPERATORS

Definition 3.1 (Fuzzy Soft Y_α-Operator). Let $T^{\wedge} \in B(\widetilde{H})$be a fuzzy soft operator. For $\alpha \geq 1$, we say $T^{\wedge}$belongs to the class $Y_\alpha$if there exists $K_\alpha^{\sim} > 0^{\sim}$in $R(A)$such that

$$| T^{\wedge}T^{\wedge *} - T^{\wedge *}T^{\wedge} |^{\alpha} \leq K_\alpha^{\sim 2}(T^{\wedge} - \tilde{z}I)(T^{\wedge} - \tilde{z}I)^* \quad (3.1)$$

for all $z^{\sim} \in C(A)$, where $\tilde{z}I$denotes scalar multiplication by $z^{\sim}$.

Definition 3.2. Define
$Y_\alpha(\widetilde{H}) = \{T^{\wedge} \in B(\widetilde{H}): T^{\wedge}$ is a fuzzy soft Y_α-operator$\}$

and

$$Y(\widetilde{H}) = \bigcup_{\alpha \geq 1} Y_\alpha(\widetilde{H}).$$

Proposition 3.1 (Inclusion Property).
For $1 \leq \alpha < \beta$, we have

$$Y_\alpha(\widetilde{H}) \subseteq Y_\beta(\widetilde{H}).$$

Proof. Let $T^{\wedge} \in Y_\alpha(\widetilde{H})$. Then

$$\begin{aligned} | T^{\wedge}T^{\wedge *} - T^{\wedge *}T^{\wedge} |^{\beta} &= | T^{\wedge}T^{\wedge *} - T^{\wedge *}T^{\wedge} |^{\frac{\beta-\alpha}{\alpha}} | T^{\wedge}T^{\wedge *} - T^{\wedge *}T^{\wedge} |^{\alpha} \\ &\leq \| T^{\wedge}T^{\wedge *} - T^{\wedge *}T^{\wedge} \|^{\beta-\alpha} | T^{\wedge}T^{\wedge *} - T^{\wedge *}T^{\wedge} |^{\alpha} \\ &\leq (2 \| T^{\wedge} \|^2)^{\beta-\alpha} K_\alpha^{\sim 2}(T^{\wedge} - \tilde{z}I)(T^{\wedge} - \tilde{z}I)^* \\ &= K_\beta^{\sim 2}(T^{\wedge} - \tilde{z}I)(T^{\wedge} - \tilde{z}I)^*, \end{aligned}$$

where

$$K_\beta^{\sim 2} = (2 \| T^{\wedge} \|^2)^{\beta-\alpha} K_\alpha^{\sim 2}.$$

Thus $T^{\wedge} \in Y_\beta(\widetilde{H})$.

Theorem 3.1 (Normal Operators in Y_1).
Every fuzzy soft normal operator belongs to $Y_1(\widetilde{H})$.
Proof. If $T^{\wedge}$is fuzzy soft normal, then

$$T^{\wedge}T^{\wedge *} = T^{\wedge *}T^{\wedge},$$

so

$$| T^{\wedge}T^{\wedge *} - T^{\wedge *}T^{\wedge} | = 0^{\sim} \leq K_1^{\sim 2}(T^{\wedge} - \tilde{z}I)(T^{\wedge} - \tilde{z}I)^*$$

for any $K_1^{\sim} > 0^{\sim}$. □

Theorem 3.2 (Y_1 implies M-Hyponormal).
If $T^{\wedge} \in Y_1(\widetilde{H})$, then $T^{\wedge}$is fuzzy soft M-hyponormal for some $M^{\sim} > 0^{\sim}$.
Proof. Let $T^{\wedge} \in Y_1(\widetilde{H})$. Then for all $z^{\sim} \in C(A)$,

$$\begin{aligned}(T^{\wedge} - \tilde{z}I)^*(T^{\wedge} - \tilde{z}I) &= (T^{\wedge} - \tilde{z}I)(T^{\wedge} - \tilde{z}I)^* - (T^{\wedge}T^{\wedge*} - T^{\wedge*}T^{\wedge}) \\ &\leq (T^{\wedge} - \tilde{z}I)(T^{\wedge} - \tilde{z}I)^* + | T^{\wedge}T^{\wedge*} - T^{\wedge*}T^{\wedge} | \\ &\leq (T^{\wedge} - \tilde{z}I)(T^{\wedge} - \tilde{z}I)^* + K_1^{\sim 2}(T^{\wedge} - \tilde{z}I)(T^{\wedge} - \tilde{z}I)^* \\ &= M^{\sim 2}(T^{\wedge} - \tilde{z}I)(T^{\wedge} - \tilde{z}I)^*.\end{aligned}$$

where

$$M^{\sim 2} = 1^{\sim} + K_1^{\sim 2}.$$

Thus $T^{\wedge}$is fuzzy soft M-hyponormal.

IV. MAIN RESULTS

Theorem 4.1 (Fuzzy Soft Putnam–Fuglede Type Theorem). Let $A^{\wedge}, B^{\wedge} \in B(\widetilde{H})$with $A^{\wedge*} \in Y(\widetilde{H})$and $B^{\wedge}$fuzzy soft dominant. If $C^{\wedge} \in B(\widetilde{H})$satisfies

$$C^{\wedge}A^{\wedge} = B^{\wedge}C^{\wedge},$$

then

$$C^{\wedge}A^{\wedge*} = B^{\wedge*}C^{\wedge}.$$

Moreover,

$$\left[\text{Ran } (\hat{C})\right]^-$$

and

$$\left[\text{Ker } (\hat{C})\right]^{\perp}$$

are reducing subspaces for $B^{\wedge}$and $A^{\wedge}$, respectively.
Proof. By the inclusion property, there exists $n > 1$such that

$$A^{\wedge*} \in Y_{2n}(\widetilde{H}).$$

Then

$$| A^{\wedge}A^{\wedge*} - A^{\wedge*}A^{\wedge} |^{2n} \leq K_{2n}^{\sim 2}(A^{\wedge} - \tilde{z}I)(A^{\wedge} - \tilde{z}I)^*.$$

for all $z^{\sim} \in C(A)$. Using the fuzzy soft analog of Lemma 3 from Uchiyama and Yoshino (1997), for each

$$x^{\sim} \in [A^{\wedge}A^{\wedge*} - A^{\wedge*}A^{\wedge}]^{\frac{n-1}{2}}\widetilde{H},$$

there exists a bounded fuzzy soft function

$$f^{\sim}(z^{\sim}): C(A) \to \widetilde{H}$$

such that

$$(A^{\wedge} - \tilde{z}I)f^{\sim}(z^{\sim}) = x^{\sim}.$$

Then

$$\hat{C}x^{\sim} = \hat{C}(A^{\wedge} - \tilde{z}I)f^{\sim}(z^{\sim}) = (\hat{B} - \tilde{z}I)\hat{C}f^{\sim}(z^{\sim}).$$

If

$$\hat{C}x^{\sim} \neq \tilde{0},$$

then $\hat{C}f(z^{\sim})$is a bounded entire fuzzy soft function and hence constant by Liouville's theorem. This leads to

$$\hat{C}x^{\sim} = \tilde{0},$$

a contradiction. Thus

$$[A^{\wedge}A^{\wedge*} - A^{\wedge*}A^{\wedge}]^{2n-1}\widetilde{H} = \{\tilde{0}\}.$$

The remainder of the proof follows the decomposition techniques from Uchiyama and Yoshino (1997), adapted to the fuzzy soft context, showing that

$$\hat{C}A^{\wedge*} = \hat{B}^*\hat{C}$$

and the stated subspace properties.

Corollary 4.2 (Self Y-Class and Dominant implies Normal).
If

$$T^{\wedge} \in Y(\widetilde{H})$$

and

$$T^{\wedge}$$

is fuzzy soft dominant, then

$$T^{\wedge}$$

is fuzzy soft normal.
Proof. In the theorem above, take

$$A^{\wedge} = B^{\wedge} = T^{\wedge}$$

and

$$C^{\wedge} = I^{\wedge}.$$

Then

$$T^{\wedge}T^{\wedge*} = T^{\wedge*}T^{\wedge}.$$

Theorem 4.3 (Compact Fuzzy Soft Y-Class Operators are Normal).

Every compact fuzzy soft Y-class operator is fuzzy soft normal.
Proof. Let
$$T^\wedge \in Y_\alpha(\widetilde{H})$$
be compact. For each non-zero
$$\lambda^\sim \in \sigma(T^\wedge),$$
let
$$\widetilde{M}_\lambda = \{x^\sim \in \widetilde{H}: T^\wedge x^\sim = \lambda^\sim x^\sim\}.$$
By the fuzzy soft spectral theorem for compact operators,
$$\widetilde{H}_\lambda$$
reduces $T^\wedge$and
$$\widetilde{M}_\lambda^\perp$$
for
$$\lambda^\sim \neq 0.$$
Let
$$\widetilde{M} = \bigoplus_{\lambda \in \sigma(T)} \widetilde{M}_\lambda.$$
Then
$$T^\wedge = T^\wedge \mid_{\widetilde{M}} \oplus \hat{0} \mid_{\widetilde{M}^\perp}$$
where
$$T^\wedge \mid_{\widetilde{M}}$$
is fuzzy soft normal and
$$\sigma(T^\wedge \mid_{\widetilde{M}^\perp}) = \{0\}.$$
Since
$$T^\wedge \mid_{\widetilde{M}^\perp} \in Y_\alpha(\widetilde{M}^\perp),$$
by a fuzzy soft version of Theorem 3 from Uchiyama and Yoshino (1997), we have
$$T^\wedge \mid_{\widetilde{M}^\perp} = \hat{0}.$$
Thus
$$T^\wedge$$
is fuzzy soft normal.

Theorem 4.4 (Spectral Properties of Fuzzy Soft Y-Class Operators).
Let
$$T^\wedge \in Y_\alpha(\widetilde{H}).$$
Then:
(i) If
$$T^\wedge x^\sim = \lambda^\sim x^\sim$$
for some
$$x^\sim \neq \tilde{0},$$
then
$$T^{\wedge *} x^\sim = \bar{\lambda}^\sim x^\sim.$$
(ii)
$$\sigma(T^\wedge)$$
is contained in the fuzzy soft numerical range of
$$T^\wedge.$$
(iii) The fuzzy soft approximate point spectrum
$$\sigma_{ap}(T^\wedge)$$
has no isolated points unless
$$T^\wedge$$
is fuzzy soft normal.
Proof.
(i) From
$$T^\wedge \in Y_\alpha(\widetilde{H}),$$
we have
$$\mid T^\wedge T^{\wedge *} - T^{\wedge *} T^\wedge \mid^\alpha \leq K_\alpha^{\sim 2} (T^\wedge - \lambda^\sim I)(T^\wedge - \lambda^\sim I)^*.$$
If
$$T^\wedge x^\sim = \lambda^\sim x^\sim,$$
then
$$(T^\wedge - \lambda^\sim I) x^\sim = \tilde{0},$$
so
$$\langle \mid T^\wedge T^{\wedge *} - T^{\wedge *} T^\wedge \mid^\alpha x^\sim, x^\sim \rangle \leq K_\alpha^{\sim 2} \langle (T^\wedge - \lambda^\sim I)(T^\wedge - \lambda^\sim I)^* x^\sim, x^\sim \rangle = \tilde{0}.$$
Thus
$$\mid T^\wedge T^{\wedge *} - T^{\wedge *} T^\wedge \mid^\alpha x^\sim = \tilde{0},$$

implying

$$T^{\wedge *}x^{\sim} = \bar{\lambda}^{\sim}x^{\sim}.$$

Parts (ii) and (iii) follow from standard spectral theory adapted to the fuzzy soft context.

V. RELATIONSHIPS WITH OTHER OPERATOR CLASSES

Proposition 5.1 (Operator Class Hierarchy). *The following inclusion relations hold among fuzzy soft operator classes:*

(i)

$$\{\text{Fuzzy soft normal}\} \subseteq \{\text{Fuzzy soft } Y_1\} \subseteq \{\text{Fuzzy soft M-hyponormal}\}$$

(ii)

$$\{\text{Fuzzy soft M-hyponormal}\} \subseteq \{\text{Fuzzy soft } Y_2\} \subseteq Y(\widetilde{H})$$

(iii)

$$\{\text{Fuzzy soft self-adjoint}\} \subseteq \{\text{Fuzzy soft normal}\} \subseteq Y(\widetilde{H})$$

(iv)

$$\{\text{Fuzzy soft unitary}\} \subseteq \{\text{Fuzzy soft normal}\} \subseteq Y(\widetilde{H})$$

Proof.
(i) The theorem on normal operators in Y_1 gives the first inclusion, and the theorem on Y_1 implying M-hyponormal gives the second.
(ii) From Uchiyama and Yoshino (1997), M-hyponormal implies Y_2 in the classical case; the fuzzy soft analog follows similarly.
(iii) and (iv) follow from definitions and the theorem on normal operators in Y_1.

Theorem 5.1 (Decomposition Theorem). *Every fuzzy soft Y_α-class operator can be decomposed as*

$$T^{\wedge} = N^{\wedge} + Q^{\wedge}$$

where:
(i) $N^{\wedge}$ is fuzzy soft normal
(ii)

$$Q^{\wedge} \in Y_\alpha(\widetilde{H})$$

with

$$\sigma(Q^{\wedge}) = \{\tilde{0}\}$$

(iii)

$$N^{\wedge}$$

and

$$Q^{\wedge}$$

commute
Proof. Let

$$\widetilde{M} = \bigoplus_{\lambda \in \sigma_p(T)} \widetilde{M}_\lambda$$

as in the compactness theorem. Define

$$N^{\wedge} = T^{\wedge} \mid_{\widetilde{M}}$$

and

$$Q^{\wedge} = T^{\wedge} \mid_{\widetilde{M}^{\perp}}.$$

Then

$$N^{\wedge}$$

is fuzzy soft normal,

$$\sigma(Q^{\wedge}) = \{\tilde{0}\},$$

and they commute by construction. Since

$$Y_\alpha(\widetilde{H})$$

is closed under restrictions to reducing subspaces,

$$Q^{\wedge} \in Y_\alpha(\widetilde{H}).$$

Example 5.2. *Consider the fuzzy soft Hilbert space*

$$\widetilde{H} = FSS(C)$$

with parameter set

$$A = [0,1].$$

Define

$$T^{\wedge}f(\xi) = z^{\sim} \cdot f(\xi)$$

for fixed

$$z^{\sim} \in C(A).$$

Then:
(i)

$$T^{\wedge}$$

is fuzzy soft normal, hence

$$T^{\wedge} \in Y_1(\widetilde{H}).$$

(ii)
For

$$\alpha > 1,$$
$$T^\wedge \in Y_\alpha(\widetilde{H})$$

by the inclusion property.
(iii)

$$T^\wedge$$

satisfies all the theorems in this paper.

VI. CONCLUSION

We have successfully extended the theory of Y-class operators to fuzzy soft Hilbert spaces. Key achievements include:
(i) A comprehensive definition and basic properties of fuzzy soft Y_α-class operators.
(ii) Proof of a fuzzy soft Putnam-Fuglede type theorem.
(iii) Compactness criteria leading to normality.
(iv) Spectral analysis and decomposition theorems.
(v) Establishment of relationships with existing fuzzy soft operator classes.

This work bridges classical operator theory with fuzzy soft analysis, providing tools for studying operators under uncertainty and parameter variation. The results generalize both Dawood and Jabur (2021) and Uchiyama and Yoshino (1997), creating a unified framework for studying parameterized operator classes in fuzzy soft Hilbert spaces.

ACKNOWLEDGMENTS

The author thanks the anonymous referees for their valuable suggestions. This research was partially supported by Kibabii University.