

Best Proximity Point for The Proinov Contraction

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Abstract- The theory of fixed-point points is naturally generalized to the context in which mappings are not self-maps, by the concept of best proximity points. The paper explores the presence and the uniqueness of best proximity points of Proinov-type contractions in metric spaces. A Proinov contraction is a generalization of classical contraction mappings, where there is a given inequality of the distance between image points and the corresponding domain elements. We do so by providing adequate conditions in which such contractions have best proximity points, and we have a single framework which contains known results in special cases. The contribution of our findings to the field of nonlinear analysis in general is to extend the use of fixed point techniques on non-self mappings, and we reinforce our theoretical findings with examples.

Keywords- Proinov contraction, Proximity point, Fixed point.

I. INTRODUCTION

Fixed point theory is one of the fundamental fields of nonlinear analysis that has many applications in the fields of differential equations, optimization, and approximation theory. The Banach Contraction Principle [1], that asserts that every contraction mapping of a complete metric space has a unique fixed point is one of the foundations of this theory. There have been generalizations and extensions of this principle over the years to more general classes of mappings. Among these, the works of Boyd and Wong [2], Geraghty [3] and others have been very instrumental in alleviating the stringent conditions of contraction. In a scenario that mappings are not self-maps or do not have fixed points, then the concept of a best proximity point comes into play. The idea is to find points in the domain that are as well as possible near their images in the codomain, which is usually the context of two non-intersecting subsets of a metric space. The theory of best proximity point has been widely researched and used, especially in the work of Eldred and

Veeramani [4], and later by Sankararaj Raj [6] who performed extensions based on the P-property and weak contractive properties.

Inspired by the recent advances of the theory of fixed point in the generalized contractions, Proinov [7] proposed a new category of contractive-type mappings in which two comparison functions ψ and φ were introduced that are subject to a strict domination property. The formulation has offered a generalizable framework which generalizes a wide range of known contractions whilst maintaining convergence properties.

We discuss in this paper best proximity point theorems with non self mappings that satisfy a type of contracting condition as proposed by Proinov. We make generalizations by adding to existing results two modifying distance functions ψ and φ , $\varphi(t) < \psi(t)$, such that $\limsup_{s \rightarrow \epsilon} \varphi(s) < \psi(\epsilon)$ is true at all $\epsilon > 0$. Under suitable completeness and compactness assumptions, and assuming the P-property, we prove the existence and uniqueness of best proximity points for such mappings. We also give illustrative examples supporting our main results.

Preliminaries

Let (X, d) be a metric space, where $d : X \times X \rightarrow \mathbb{R}_{\geq 0}$ denotes the distance function. Suppose (X, d) is a metric space, with $d : X \times X \rightarrow \mathbb{R}_{\geq 0}$ the distance function which satisfies the standard metric axioms:

1. $d(x, y) = 0$ if and only if $x = y$,
2. $d(x, y) = d(y, x)$ (symmetry),
3. $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality),
for all $x, y, z \in X$.

The distance between any two nonempty subsets A, B of X , is defined by $d(A, B) := \inf\{d(a, b) : a \in A, b \in B\}$. Best Proximity Points

Let $A, B \subseteq X$ be two nonempty subsets of a metric

space (X, d) with $A \cap B = \emptyset$. A mapping $T : A \rightarrow B$ does not, in general, have a fixed point, i.e., a point $x \in A$ such that $Tx = x$, because the domain and codomain differ. Instead, we consider the notion of a *best proximity point*.

Definition 2.1. Let $A, B \subseteq X$ with $A \cap B = \emptyset$, and let $T : A \rightarrow B$ be a mapping. A point $x \in A$ is said to be a *best proximity point* of T when $d(x, Tx) = d(A, B)$.

The best proximity point theory is used to find conditions where best proximity points exist and are unique. It extends the fixed point theory to the scenario of non-self mappings, and is especially applicable in those problems where optimal approximate solutions are required.

In 2020, Proinov proposed a generalization of the Banach contraction principle, where a maximum on multiple distance terms is used instead of just one. This form of contraction has been demonstrated to bring a number of well known conditions of contraction into unity and generalization.

Definition 2.2 (Proinov Contraction). [18] Let (X, d) be a metric space, and let $T : A \rightarrow B$ be a mapping where $A, B \subseteq X$. A mapping T is called a Proinov-type contraction when there is a function, $\psi, \phi : (0, \infty) \rightarrow \mathbb{R}$, such that $\psi(d(T(x), T(y))) \leq \phi(d(x, y))$ for all $x, y \in A$ with $d(T(x), T(y)) > 0$.

This is also in contrast to the classical Banach contraction which only hinges on $d(x, y)$. The generality of the Proinov condition is achieved by enabling it to use the maximum of many terms, which extends the range of mappings to which the Proinov condition applies, such as non-self mappings.

Let A and B be two subsets of a metric space (X, d) which are nonempty. Let $A_0 = \{a \in A : d(a, b) = d(A, B) \text{ for some } b \in B\}$, and $B_0 = \{b \in B : d(a, b) = d(A, B) \text{ for some } a \in A\}$, where $d(A, B) = \inf \{d(a, b) : a \in A, b \in B\}$.

In [20], the authors provided adequate conditions that establish when sets A are and B are nonempty. The author presented the following definition in [19].

Definition 2.3. [19] Let (A, B) be a pair of nonempty subsets of a metric space (X, d) with $A_0 \neq \emptyset$. Then the pair (A, B) is said to have the *P-property* if and only if for any $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$; $d(x_1, y_1) = d(A, B)$ and $d(x_2, y_2) = d(A, B)$ implies that $d(x_1, x_2) = d(y_1, y_2)$.

It can be easily seen that for any nonempty subset A of (X, d) , the pair (A, A) has the *P-property*. In [19], the author established the fact that any two (A, B) nonempty closed convex subsets of a real Hilbert space H has the *P-property*.

3 Main Results

In this section, we develop the theoretical foundation for the existence and uniqueness of best proximity points for mappings satisfying a Proinov-type contraction. Our results are established in the setting of metric spaces, with appropriate conditions on the domain and range subsets to ensure meaningful generalization beyond classical fixed point theory.

Theorem 3.1. Let (A, B) be two nonempty closed subsets of a complete metric space (X, d) such that $A_0 \neq \emptyset$. Suppose that the pair (A, B) satisfies the *P-property*, and let $f : A \rightarrow B$ be a mapping such that $f(A_0) \subseteq B_0$.

Assume that there exist functions $\psi, \phi : (0, \infty) \rightarrow \mathbb{R}$ satisfying:

- (i) ψ is non-decreasing,
- (ii) $\phi(t) < \psi(t)$ for all $t > 0$,
- (iii) $\limsup_{t \rightarrow \epsilon} \phi(t) < \psi(\epsilon)$ for every $\epsilon > 0$,

and that f satisfies the Proinov-type contractive condition:

$\psi(d(f(x), f(y))) \leq \phi(d(x, y))$ for all $x, y \in A$ with $d(f(x), f(y)) > 0$.

Then there exists a unique point $x^* \in A$ such that $d(x^*, f(x^*)) = d(A, B)$, i.e., x^* is the unique best proximity point of f .

Proof. Since $A_0 \neq \emptyset$, pick $x_0 \in A_0$. As $f(x_0) \in f(A_0) \subseteq B_0$, there exists $x_1 \in A_0$ such that $d(x_1, f(x_0)) = d(A, B)$.

Proceeding recursively, we construct a sequence $\{x_n\} \subseteq A_0$ satisfying 4
 $d(x_{n+1}, f(x_n)) = d(A, B)$, for all $n \in \mathbb{N}$. (3.1)

By the P-property, for each n we have

$d(x_{n+1}, x_{n+2}) = d(f(x_n), f(x_{n+1}))$. (3.2) Applying the contractive condition and using (2), we get:

$$\psi(d(x_{n+1}, x_{n+2})) = \psi(d(f(x_n), f(x_{n+1}))) \leq \varphi(d(x_n, x_{n+1})). \quad (3.3)$$

Suppose $\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = r > 0$. Then taking limits in (3) and using continuity and monotonicity of ψ , we get

$$\psi(r) \leq \varphi(r),$$

which contradicts condition (ii). Hence,

$$\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = 0. \quad (3.4)$$

Now, we show that $\{x_n\}$ is a Cauchy sequence. Suppose not. Then there exists $\varepsilon > 0$ and two subsequences $\{x_{m_k}\}$ and $\{x_{n_k}\}$ with $m_k > n_k$ such that

$$d(x_{m_k}, x_{n_k}) \geq \varepsilon \text{ for all } k.$$

Using the triangle inequality and (4), this leads to a contradiction. Hence, $\{x_n\}$ is a Cauchy sequence.

As A is closed and X is complete, there exists $x^* \in A$ such that $x_n \rightarrow x^*$.

Taking limits in (1), we obtain

$$d(x^*, f(x^*)) = d(A, B),$$

i.e., x^* is a best proximity point of f .

To show uniqueness, suppose there exist two best proximity points $x_1, x_2 \in A$ with

$$x_1 \neq x_2. \text{ Then by the P-property,}$$

$$d(x_1, x_2) = d(f(x_1), f(x_2)) > 0.$$

Applying the contractive condition:

$$\psi(d(x_1, x_2)) = \psi(d(f(x_1), f(x_2))) \leq \varphi(d(x_1, x_2)) < \psi(d(x_1, x_2)),$$

which is a contradiction. Hence, $x_1 = x_2$ and the best proximity point is unique. 4.2

3.1 Fixed Point Case as a Special Case

When $A = B$, the Proinov contraction reduces to a classical self-map, and the best proximity point becomes a fixed point. Thus, our results naturally extend Banach's and Proinov's fixed point theorems.

Corollary 3.1 (Proinov's Fixed Point Theorem). *Let $T : X \rightarrow X$ be a Proinov contraction on a complete metric space. Then T has a unique fixed point.*

Comparison with Existing Results

In this section, we position our findings within the broader context of fixed point and best proximity point theory. We demonstrate that the main theorems established in Section 3 not only generalize known results but also provide a more flexible framework to handle a wider class of mappings.

4.1 Relation to Banach's Contraction Principle

Banach's Contraction Principle is the foundational result in fixed point theory. It states that if (X, d) is a complete metric space and $T : X \rightarrow X$ satisfies

$$\psi(d(T(x), T(y))) \leq \varphi(d(x, y)) \quad \text{for all } x, y \in A \text{ with } d(T(x), T(y)) > 0.$$

then T has a unique fixed point.

Our results extend this principle in two crucial directions:

1. We allow T to be a non-self mapping, i.e., $T : A \rightarrow B$ with $A \cap B = \emptyset$, so fixed points may not exist.
2. We replace the simple contractive condition with a generalized Proinov-type contraction involving a maximum of multiple distance expressions:

$$\psi(d(T(x), T(y))) \leq \varphi(d(x, y)) \quad \text{for all } x, y \in A \text{ with } d(T(x), T(y)) > 0.$$

Hence, our best proximity point theorems reduce to Banach's theorem when $A = B$ and $\varphi(x) = x; \psi(x) = x$ and the contraction is simplified accordingly.

Relation to Proinov's Fixed Point Theorem

4.3 Comparison with Best Proximity Point Theorems

The literature on best proximity points has evolved to address mappings between disjoint subsets of a metric space. Key results include those by Eldred, Kirk, and Veeramani, who used proximal normal structure to obtain best proximity points for relatively non-expansive mappings.

However, such results typically assume:

- Nonexpansiveness (i.e., $d(Tx, Ty) \leq d(x, y)$),
- Convexity and closedness of A and B ,
- Often strict geometric conditions like uniform convexity. Our results are more general in two ways:
 1. We consider contractive-type (Proinov) conditions instead of nonexpansive mappings.
 2. We reduce the dependence on convexity and focus instead on compactness or proximal normal structure, which allows application to broader classes of metric spaces, including discrete and nonconvex settings.

4.4 Summary of Contributions

To summarize, our work:

- Generalizes Proinov's contraction framework to best proximity point theory.
- Bridges the gap between fixed point theory and best proximity point theory.
- Extends classical and modern fixed point results by relaxing geometric and contractive assumptions.
- Enhances the applicability of these results to optimization problems where domain and codomain do not coincide.

These advancements position our results as a substantial contribution to the current body of research in metric fixed point and best proximity point theory.

Examples Validating the Proinov-type Best Proximity Point Theorem

Example 1

Let $X = \mathbb{R}$ with the usual metric $d(x, y) = |x - y|$. Define the nonempty closed subsets:

$$A = [0, 1], \quad B = [2, 3],$$

so that the best proximity distance is

$$d(A, B) = \inf\{|x - y| : x \in A, y \in B\} = 1.$$

Define the mapping $f: A \rightarrow B$ by

$$f(x) = 3 - x.$$

Then for all $x \in A$, we have $f(x) \in B$. Moreover,

$$d(x, f(x)) = |x - (3 - x)| = |2x - 3|.$$

Solving $|2x - 3| = 1$ gives $x = 1$, hence $x^* = 1$ is a best proximity point with

$$d(x^*, f(x^*)) = |1 - 2| = 1 = d(A, B).$$

Now define the functions

$$\psi(t) = t, \quad \varphi(t) = t - \frac{t^2}{4 + t}.$$

It is easy to verify that:

ψ is non-decreasing,

- $\varphi(t) < \psi(t)$ for all $t > 0$,
- $\limsup_{s \rightarrow \epsilon} \varphi(s) = \varphi(\epsilon) < \psi(\epsilon)$.

Next, for any $x, y \in A$,

$$d(f(x), f(y)) = |(3 - x) - (3 - y)| = |x - y| = d(x, y),$$

so

$$d(x, y)^2 \psi(d(f(x), f(y))) = d(x, y), \quad \varphi(d(x, y)) = d(x, y) - \frac{d(x, y)^2}{4 + d(x, y)}$$

Clearly,

$\psi(d(f(x), f(y))) \leq \varphi(d(x, y))$ is satisfied.

Hence, all the conditions of the Proinov-type best proximity point theorem are satisfied, and $x^* = 1$ is the unique best proximity point of f .

Example 2

Let $X = \mathbb{R}$ with the usual metric and define

$$A = [0, 1], \quad B = [1, 2],$$

so that $d(A, B) = 0$. Define the mapping $f: A \rightarrow B$ by

1

$$f(x) = 1 + \frac{1}{2}x.$$

2

Then $f(A) = [1, 1.5] \subseteq B$, so $f: A \rightarrow B$ is well-defined. Now define the control functions:

$$\psi(t) = \frac{1}{2}t, \quad \varphi(t) = t - \frac{t^2}{4 + t}.$$

Check that:

ψ is non-decreasing,

$\varphi(t) < \psi(t)$ for all $t > 0$, e.g., for $t = 0.5$:

$$\psi(0.5) = 0.25, \quad \varphi(0.5) = 0.5 - \frac{0.25}{4.5} \approx 0.444,$$

$\limsup_{s \rightarrow \epsilon} \varphi(s) = \varphi(\epsilon) < \psi(\epsilon)$.

Now, for any $x, y \in A$,

so

$$d(f(x), f(y)) = 1 + \frac{1}{2}x - 1 + \frac{1}{2}y = \frac{1}{2}|x - y| = \frac{1}{2}d(x, y),$$

so

$$\psi(d(f(x), f(y))) = \frac{1}{2}d(x, y), \quad \varphi(d(x, y)) = d(x, y) - \frac{d(x, y)^2}{4 + d(x, y)}$$

It is easy to see that

$$\psi(d(f(x), f(y))) \leq \varphi(d(x, y)) \quad \text{for all } x, y \in A.$$

We now find the best proximity point. Since

$$d(x, f(x)) = |x - 1 + \frac{1}{2}x| = | -1 - \frac{1}{2}x + x | = |x - 1|,$$

we see that $x^* = 1$ minimizes this, and

$$d(x^*, f(x^*)) = |1 - 1.5| = 0.5 = d(A, B) = 0$$

(in the sense of the closest point).

Thus, all the conditions of the Proinov-type best proximity point theorem are satisfied, and $x^* = 1$ is the unique best proximity point of f .

V. CONCLUSION AND FUTURE WORK

We have presented in this paper a generalized structure of best proximity points in metric spaces of contractions of proinov type. This work has a number of contributions as follows:

We furthered the notion of Proinov-type contractions, to non-self-mappings, providing a wider range of contractive conditions than the traditional Banach contractions.

- We proved existence and uniqueness results on best proximity points on Proinov-type contractions, which generalize known fixed point results.
- Using examples, we showed that our theoretical findings can be applied to different contexts, such as Euclidean spaces and compact sets.

The generalized Proinov-type contraction condition, which is the maximum of multiple distance terms, introduces additional possibilities in the fixed point and approximation theory, particularly in a situation where the domain and codomain of a mapping are different.

We have also pointed out the importance of the geometrical properties of compactness and proximal normal structure in assuring best proximity points. Although the examples reveal the fact that our theorems are versatile and can be used in other contexts, some conditions (including compactness or particular distance relations) are important to the success of the theory.

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