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Application of the Homotopy Perturbation Method to Selected Nonlinear and Fractional Differential Equations with Comparative Analysis

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Abstract- In this study, the Homotopy Perturbation Method (HPM) is applied to obtain approximate analytical solutions of selected nonlinear and fractional differential equations. The method is first introduced and then implemented on a nonlinear ordinary differential equation, Burgers' equation, and a fractional-order logistic model. The solutions are expressed in the form of rapidly convergent series without the need for discretization, linearization, or the presence of small parameters. For the nonlinear ordinary differential equation, the HPM solution is shown to converge to the exact solution with high accuracy. In the case of Burgers' equation, the method effectively captures both nonlinear convective and diffusive effects, demonstrating its capability in handling nonlinear partial differential equations. Furthermore, the application of HPM to the fractional logistic equation highlights its effectiveness in solving fractional-order systems, where the solutions exhibit memory-dependent dynamics characterized by fractional powers of time. Comparative analysis indicates that HPM provides accurate results with reduced computational effort when compared to other semi-analytical methods. Graphical results further validate the convergence and accuracy of the method. The findings confirm that HPM is a reliable, efficient, and versatile technique for solving a wide range of nonlinear and fractional differential equations.

Index Terms- Homotopy Perturbation Method; nonlinear differential equation; Burgers' equation; fractional logistic equation; Caputo fractional derivative.

I. INTRODUCTION

Nonlinear differential equations arise naturally in the modeling of many real-world phenomena in science, engineering, and applied mathematics, including fluid mechanics, population dynamics, epidemiology, and heat transfer processes. Nonlinear differential equation models are widely used in epidemiology to study disease dynamics and control strategies, including optimal control formulations that incorporate factors such as public awareness (enlightenment) and drug effectiveness [1,2]. Due to the inherent nonlinearity of such problems, obtaining exact analytical solutions is often difficult or, in many cases, impossible [3]. Consequently, sconsiderable attention has been devoted to the development of efficient analytical and semi-analytical techniques for approximating solutions to these equations. The computation of epidemiological thresholds such as the basic reproduction number in nonlinear models using the next-generation matrix further underscores the significance of efficient analytical methods [4].

Among the available methods, the Homotopy Perturbation Method (HPM), introduced by [5], has gained significant popularity due to its simplicity, rapid convergence, and ability to handle strong nonlinearities without requiring small perturbation parameters. The method combines the classical perturbation techniques with the concept of homotopy from topology [6], thereby constructing a continuous transformation from a simple problem to the original nonlinear problem. Unlike traditional perturbation methods, HPM does not depend on the presence of a small parameter, which enhances its applicability to a wider class of problems [7]. Alongside analytical and semi-analytical techniques for nonlinear and fractional differential equations, classical numerical methods such as the Jacobi and Gauss–Seidel iterative schemes remain fundamental tools for solving linear systems [8]. Over the years, HPM has been successfully applied to various types of differential equations, including ordinary differential equations, partial differential equations, and integral equations. For instance, it has been effectively utilized in solving nonlinear oscillatory systems, fluid flow models, and reaction-diffusion equations [9,10]. Its efficiency and accuracy have also been demonstrated through comparisons with other semi-analytical methods such as the Adomian Decomposition Method (ADM) and the Variational Iteration Method (VIM), where HPM often yields rapidly convergent series solutions with minimal computational effort [11, 5].

In recent years, there has been increasing interest in the study of fractional differential equations, which generalize classical integer-order models by incorporating memory and hereditary properties of various materials and processes [12,13]. These equations have proven to be more suitable for describing complex systems in fields such as viscoelasticity, control theory, and epidemiology [14,15]. However, the non-local nature of fractional derivatives presents additional challenges in obtaining analytical solutions, thereby necessitating the use of efficient approximation techniques. The increasing use of Atangana–Baleanu fractional-order models in epidemiology, such as those developed for norovirus [16] and hepatitis B virus transmission dynamics [17], further motivates the application of semi-analytical techniques like the Homotopy Perturbation Method for obtaining approximate solutions. The application of HPM to fractional differential equations has shown promising results, particularly when combined with definitions such as the Caputo fractional derivative, which allows for the incorporation of physically meaningful initial conditions. Fractional-order modeling has also been extended to macroeconomic systems, where memory effects are incorporated using Atangana–Baleanu fractional derivatives, as demonstrated in studies of inflation dynamics and consumer price index forecasting [18,19,20]. Several studies have demonstrated that HPM can produce accurate approximate solutions for fractional-order models with relatively few iterations [21]. [22] applies Ji-Huan He’s homotopy perturbation method with Jacobi elliptic initial approximations to solve strongly nonlinear second-order oscillatory differential equations, showing highly accurate long-term agreement with numerical solutions despite unclear theoretical justification. [23] shows that the homotopy perturbation method is a special case of the more general homotopy analysis method, which uses an auxiliary parameter h to control convergence without a precise initial guess. [24] applies the homotopy perturbation method to solve the nonlinear two-dimensional wave equation, yielding a rapidly convergent series solution that handles nonhomogeneous terms via self-canceling “noise” components and demonstrates high accuracy and reliability compared to existing methods. [25] presents the homotopy-perturbation method as an efficient and broadly applicable approach for solving nonlinear reaction–diffusion systems, outperforming the Adomian decomposition method while avoiding its computational difficulties. [26] applies the Laplace homotopy perturbation method to solve one-dimensional nonhomogeneous variable-coefficient PDEs, achieving accurate, efficient solutions that agree closely with exact and numerical methods while requiring less computation than existing approaches. [27] modifies the Mohand transform and combines it with the homotopy perturbation method to solve the fractional Newell–Whitehead–Segel equation in the Caputo sense, using HPM to handle nonlinear terms beyond the transform’s linear limitations. [28] analyzes the homotopy perturbation method for solving fractional PDEs, establishes a unified convergence theorem, and validates it through applications to the fractional Burger–Poisson equation and challenging fractional boundary value problems.

Motivated by these developments, this study aims to apply the Homotopy Perturbation Method to selected nonlinear and fractional differential equations, including ordinary differential equations, partial differential equations, and fractional-order models. The objective is to demonstrate the effectiveness, accuracy, and convergence behavior of HPM across different classes of problems. Additionally, comparisons are made with the analytical and numerical solutions to validate the results obtained.
Despite these advancements, many existing studies focus either on integer-order or fractional-order problems in isolation, with limited work providing a

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unified treatment that combines detailed analytical solutions and graphical validation across multiple classes of equations. In particular, there is a need for studies that systematically demonstrate the convergence, accuracy, and applicability of HPM to both nonlinear and fractional models within a single framework. This study aims to address this gap by applying HPM to selected nonlinear and fractional differential equations and validating the results through analytical derivations and numerical simulations. The remainder of this paper is organized as follows: section 2 presents the basic formulation of the Homotopy Perturbation Method. Section 3 applies the method to selected non-linear ordinary and partial differential equations, and also a fractional order logistic model. Section 4 focuses on the numerical simulation, while section 5 discusses the results, provides comparisons, and concludes the study.

II. METHODOLOGY

This section presents the fundamental formulation of the Homotopy Perturbation Method (HPM) and its application to nonlinear and fractional differential equations considered in this study. The method is systematically applied to three classes of problems: a non-linear ordinary differential equation, Burgers' equation, and a fractional logistic equation. For each case, the homotopy is constructed, the series solution is derived up to a finite number of terms, and the approximate solution is obtained. To validate the analytical results, numerical simulations are carried out using MATLAB. The solutions obtained are analyzed in terms of convergence, accuracy, and physical consistency through graphical representations.

2.1 Overview of the Homotopy Perturbation Method

The Homotopy Perturbation Method (HPM) is a semi-analytical technique used to solve linear and non-linear differential equations [29]. The homotopy perturbation method (HPM) was first proposed by a Chinese mathematician, He in 1998 [30]. In HPM, a difficult problem is transformed into a simpler one through a homotopy, and then a perturbation parameter is used to continuously deform the solution of the simple problem into the solution of the original problem [31,32].

The basic idea of the homotopy perturbation method is to introduce a homotopy parameter, say p, which takes values from 0 to 1. For p=0, the system of equations usually reduces to a sufficiently simplified form, which normally admits a rather simple solution [33]. As p gradually increases to 1, the system goes through a sequence of "deformation", the solution of each of which is "close" to that at the previous stage of "deformation". Eventually at p=1, the system takes the original form of the equation, and the final stage of "deformation" gives the desired solution [34].

Suppose we have a nonlinear differential equation:

$$A(u) - f(r) = 0, r \in \Omega . \quad (2.1)$$

With the boundary conditions:

$$B\left(u, \quad \frac{\partial u}{\partial n}\right) = 0, r \in \Gamma. \quad (2.2)$$

Where A is a general differential operator, f(r)is a known analytic function, u is the unknown function, B is a boundary operator, and Γ is the boundary of the domain Ω.

The operator A, can be split into

A=L+N

where,L is the linear part,and N is the nonlinear part. The homotopy perturbation structure H(v,p) is constructed as follows.

$$H(v,p) = (1-p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0, \quad [30]$$

$$H(v,p) = (1-p)[L(v) - L(u_0)] + p[L(v) + N(v) - f(r)] = 0,$$

$$H(v,p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)] = 0.$$

Where,
$v(r,p): \Omega \times [0,1] \rightarrow R, r \in \Omega$ and $p \in [0,1]$ is an embedding (perturbation) parameter, u_0 is the first approximation which satisfies the boundary conditions.

For p=0, and p=1, we have:

$$H(v,0) = L(v) - L(u_0) = 0$$
$$H(v,1) = L(v) + N(v) - f(r) = 0$$

The process of changes in p from 0 to 1 is that of changing from $u_0(r)$ to $u(r)$. The basic assumption is that the solution can be expressed as a power series in p:

$$v = v_0 + pv_1 + p^2 v_2 + \cdots$$

The approximate solution of equation (2.1) therefore, can be readily obtained as:

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \cdots \text{ [30].}$$

2.2 Application to Nonlinear Ordinary Differential Equations

For non-linear ordinary differential equations (ODEs), the method follows a systematic procedure. First, the equation is rewritten in operator form and decomposed into linear and nonlinear components. A homotopy is then constructed, and a series solution is assumed.

By substituting the series expansion into the homotopy equation and equating coefficients of like powers of p, a sequence of linear differential equations is obtained. These equations are solved recursively using the given initial or boundary conditions. This approach avoids linearization or discretization and provides an analytical approximation in series form [5].

2.3 Application to Nonlinear Partial Differential Equations

For non-linear partial differential equations (PDEs), such as Burgers' equation, the HPM procedure is extended by considering appropriate initial and boundary conditions. The solution is again expressed as a series in terms of the embedding parameter p.

After substituting the series into the governing PDE, terms are grouped according to powers of p, leading to a hierarchy of linear PDEs. These equations can be solved sequentially using classical methods. The approximate solution is then obtained by summing the series terms. This approach has been successfully applied to various nonlinear PDEs, including fluid flow and heat transfer problems [9].

2.4 Application to Fractional Differential Equations

To extend HPM to fractional differential equations, the fractional derivative is defined in the Caputo sense due to its suitability for physical problems with initial conditions [13]. A general fractional differential equation can be written as:

$$D_t^{\alpha} u(t) + N\big(u(t)\big) = g(t), 0 < \alpha \leq 1$$

Where $D_t^{\alpha} u(t)$ denotes the Caputo fractional derivative of u(t) of order α. The homotopy construction follows a similar approach as in the integer-order case, and the solution is assumed as a power series in p. The fractional derivative is applied term-by-term, and properties of fractional calculus, including the Gamma function, are utilized in solving the resulting equations.

By equating coefficients of like powers of p, a system of fractional differential equations is obtained and solved iteratively. Studies have shown that HPM is highly effective for fractional problems and yields accurate approximations with relatively low computational effort [21,14].

2.5 Convergence and Accuracy Considerations

The convergence of the HPM series solution depends on the nature of the nonlinear operator and the choice of the initial approximation. In many practical problems, the series converges rapidly, and only a few terms are required to achieve high accuracy. The approximate solutions obtained are compared, where possible, with exact or numerical solutions to validate the effectiveness of the method. Error analysis is carried out by computing the difference between the approximate and exact (or numerical) solutions. Graphical representations are also used to illustrate the convergence behavior and accuracy of the method.

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2.6 Computational Implementation

All computations in this study can be implemented using symbolic and numerical software such as MATLAB or Python. These tools facilitate the evaluation of series solutions, computation of fractional derivatives, and graphical visualization of results. The iterative structure of HPM makes it particularly suitable for such computational platforms.

In summary, the Homotopy Perturbation Method provides a unified and efficient framework for solving a wide range of nonlinear and fractional differential equations without requiring discretization, linearization, or small parameters. Its flexibility and accuracy make it a valuable tool in applied mathematics and related fields [29].

III. PROBLEM STATEMENT

1. Consider the non-linear ODE

$$\frac{dy}{dt} + y^2 = 0, \; y(0) = 1$$

We construct the following homotopy:

$$\frac{dy}{dt} + p\,y^2 = 0$$

Where p∈[0,1]

Assume a series solution:

$$y = y_0 + py_1 + p^2 y_2 + \cdots$$

Which can be substituted into the homotopy equation as follows:

$$\frac{d}{dt}(y_0 + py_1 + \cdots) + p(y_0 + py_1 + \cdots)^2 = 0$$

Equate Powers of p;

Order p^0:

$$\frac{dy_0}{dt} = 0$$

Integrating and applying the initial condition, we have:

$$y_0 = 1$$

Order p^1:

$$\frac{dy_1}{dt} + y_0^2 = 0$$

$$\frac{dy_1}{dt} + 1 = 0$$

Integrating we have:

$$y_1 = -t$$

Order p^2:

$$\frac{dy_2}{dt} + 2y_0 y_1 = 0$$

$$\frac{dy_2}{dt} + 2(1)(-t) = 0$$

$$\frac{dy_2}{dt} - 2t = 0$$

Integrate to obtain:

$$y_2 = t^2$$

The approximate solution is:

$$y = 1 - t + t^2 - t^3 + \cdots$$

This is a series expansion of:

$$y = \frac{1}{1+t}$$

Therefore, the HPM recovers the exact solution: $y(t) = \frac{1}{1+t}$

2. Next, we consider the one-dimensional Burgers' Equation:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = v\frac{\partial^2 u}{\partial x^2}, \; t > 0$$

with initial condition:

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$u(x,0) = \sin\ x$

where ν is the viscosity coefficient.

Using the Homotopy Perturbation Method, we introduce the embedding parameter $p \in [0,1]$:

$$\frac{\partial u}{\partial t} + p\left(u\frac{\partial u}{\partial x} - \nu\frac{\partial^2 u}{\partial x^2}\right) = 0$$

Then we assume a series solution

$$u = u_0 + pu_1 + p^2u_2 + p^3u_3 + \cdots$$

Substitute the above series solution into the homotopy equation:

$$\frac{\partial}{\partial t}(u_0 + pu_1 + p^2u_2 + \cdots) + p[(u_0 + pu_1 + \cdots)(u_{0x} + pu_{1x} + \cdots) - \nu(u_{0xx} + pu_{1xx} + \cdots)] = 0$$

Equate Powers of p:

Order p^0:

$$\frac{\partial u_0}{\partial t} = 0$$

Integrating and applying the initial condition, we have:

$$u_0(x,t) = \sin\ x$$

Order p^1:

$$\frac{\partial u_1}{\partial t} + u_0\frac{\partial u_0}{\partial x} - \nu\frac{\partial^2 u_0}{\partial x^2} = 0$$

Computing the values of the terms, we have:

$u_0 = \sin\ x$

Taking the partial derivative with respect to $x,\ u_{0x} = \cos\ x$

The second derivative, $u_{0xx} = -\sin\ x$

Substituting gives:

$$\frac{\partial u_1}{\partial t} + \sin x\cos\ x + \nu\sin\ x = 0$$

$$\Rightarrow \frac{\partial u_1}{\partial t} = -\sin x\cos\ x - \nu\sin x$$

Integrating with respect to t:

$u_1 = -t(\sin\ x\cos\ x + \nu\sin\ x)$

Order p^2:

$$\frac{\partial u_2}{\partial t} + (u_0u_{1x} + u_1u_{0x}) - \nu u_{1xx} = 0$$

$$\frac{\partial u_2}{\partial t} = -(u_0u_{1x} + u_1u_{0x}) + \nu u_{1xx}$$

From what was obtained earlier:

$u_0 = \sin\ x$

$u_{0x} = \cos\ x$

$u_1 = -t(\sin\ x\cos\ x + \nu\sin\ x)$

$u_{1x} = -t(\cos^2 x - \sin^2 x + \nu\cos\ x)$

$u_{1xx} = t(4\sin\ x\cos\ x + \nu\sin\ x)$

Let us compute each of the terms in (∂u_2)/∂t separately.

Term A: u_0u_{1x}

$u_0u_{1x} = \sin\ x \cdot [-t(\cos^2 x - \sin^2 x + \nu\cos\ x)]$

$= -t\sin\ x\cos^2 x + t\sin\ x\sin^2 x - t\nu\sin\ x\cos\ x$

Term B: u_1u_{0x}

$u_1u_{0x} = [-t(\sin\ x\cos\ x + \nu\sin\ x)]\cos\ x$

$= -t\sin\ x\cos^2 x - t\nu\sin\ x\cos\ x$

Term C: νu_{1xx}

$\nu u_{1xx} = \nu t(4\sin\ x\cos\ x + \nu\sin\ x)$

$= 4\nu t\sin\ x\cos\ x + \nu^2 t\sin\ x$

Now, let's add the first two terms i.e. $u_0u_{1x} + u_1u_{0x}$.

$$= -t\sin\ x\cos^2 x + t\sin\ x\sin^2 x - t\nu\sin\ x\cos\ x + [-t\sin\ x\cos^2 x - t\nu\sin\ x\cos\ x]$$

$$\Rightarrow u_0u_{1x} + u_1u_{0x} = -2t\sin\ x\cos^2 x + t\sin\ x\sin^2 x - 2t\nu\sin\ x\cos\ x$$

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Applying the negative sign:

$$-(u_0 u_{1x} + u_1 u_{0x}) = 2t\sin x\cos^2 x - t\sin x\sin^2 x + 2tv\sin x\cos x$$

Next, we add term C; vu_{1xx}:

i. e. $-(u_0 u_{1x} + u_1 u_{0x}) + vu_{1xx} =$

$$2t\sin x\cos^2 x - t\sin x\sin^2 x + 2tv\sin x\cos x + 4vt\sin x\cos x + v^2 t\sin x$$

$$\Rightarrow \frac{\partial u_2}{\partial t} = 2t\sin x\cos^2 x - t\sin x\sin^2 x + 6tv\sin x\cos x + v^2 t\sin x$$

Using the identity
$\sin^2 x + \cos^2 x = 1 \Rightarrow \sin^2 x = 1 - \cos^2 x$,
the term $-\sin x\sin^2 x$ can be simplified to:
$-\sin x(1 - \cos^2 x) = -\sin x + \sin x\cos^2 x$,
then substituting back we have:

$$\frac{\partial u_2}{\partial t} = 2t\sin x\cos^2 x - t\sin x + t\sin x\cos^2 x + 6tv\sin x\cos x + v^2 t\sin x$$

$$= 3t\sin x\cos^2 x - t\sin x + 6tv\sin x\cos x + v^2 t\sin x$$

$$= t[3\sin x\cos^2 x - \sin x + 6v\sin x\cos x + v^2\sin x]$$

Integrating gives:

$$u_2 = \frac{t^2}{2}[3\sin x\cos^2 x - \sin x + 6v\sin x\cos x + v^2\sin x]$$

Setting $p = 1$, the approximate solution is:

$$u(x,t) = u_0 + u_1 + u_2 + \cdots$$

$$u(x,t) = \sin x - t(\sin x\cos x + v\sin x) + \frac{t^2}{2}[3\sin x\cos^2 x - \sin x + 6v\sin x\cos x + v^2\sin x] + \cdots$$

$$u(x,t) = \sin x - t\sin x(\cos x + v) + \frac{t^2}{2}\sin x[3\cos^2 x - 1 + 6v\cos x + v^2] + \cdots$$

The approximate solution obtained via the homotopy perturbation method demonstrates rapid convergence and effectively captures both the nonlinear convective (uu_x) and diffusive terms (vu_{xx}) of Burgers' equation.

3. Consider the fractional-order logistic equation given by:

$$D_t^{\alpha} x(t) = rx(t)(1 - x(t)),\ 0 < \alpha \leq 1,$$

With the initial condition: $x(0) = x_0$. Where D_t^{α} denotes the Caputo fractional derivative of order α, and $r > 0$ is the growth rate. Fractional differential equations of this type are widely used in modeling systems with memory and hereditary properties [13,14].

The Caputo fractional derivative is defined as:

$$D_t^{\alpha} x(t) = \left(\frac{1}{\Gamma(1-\alpha)}\right)\int_0^t \left[\frac{x\prime(\tau)}{(t-\tau)^{\alpha}}\right] d\tau$$

which allows the incorporation of classical initial conditions, making it suitable for physical and biological models [13].

Applying the Homotopy Perturbation Method (HPM) [5], we construct the following homotopy:

$$D_t^{\alpha} x(t) - prx(t)(1 - x(t)) = 0,$$

where $0 \leq p \leq 1$, is the embedding parameter.
We assume the solution in the form of a power series:

$$x(t) = x_0 + px_1(t) + p^2 x_2(t) + p^3 x_3(t) + \cdots$$

Substituting into the homotopy equation gives:

$$D_t^{\alpha}(x_0 + px_1 + p^2 x_2 + \cdots) - pr(x_0 + px_1 + p^2 x_2 + \cdots)(1 - x_0 - px_1 - p^2 x_2 - \cdots) = 0.$$

Expanding the nonlinear term we have:

$$x(1-x) = x_0 - x_0^2 - px_0x_1 - p^2x_0x_2 + px_1 - px_0x_1 - p^2x_1^2 - p^3x_1x_2 + p^2x_2 - p^2x_0x_2 - p^3x_1x_2 - p^4x_2^2 - \cdots$$

$$\Rightarrow x(1-x) = x_0(1-x_0) + px_1(1-2x_0) + p^2x_2[(1-2x_0) - x_1^2] + \cdots$$

$$\text{Thus } D_t^{\alpha}(x_0 + px_1 + p^2x_2 + \cdots) - pr\{x_0(1-x_0) + px_1(1-2x_0) + p^2[x_2(1-2x_0) - x_1^2] + \cdots\} = 0.$$

Next, we equate powers of p:
Order p^0:

$$D_t^{\alpha} x_0 = 0$$

$x_0 = \text{constant}$

Order p^1:

$$D_t^\alpha x_1(t) = rx_0(1 - x_0)$$

Applying the fractional integration operator J_t^α; the inverse operator of D_t^α, we obtain:

$$x_1(t) = J_t^\alpha[rx_0(1 - x_0)]$$
$$= \frac{t^\alpha}{\Gamma(\alpha+1)}[rx_0(1 - x_0)]$$

Order p^2:

$$D_t^\alpha x_2(t) = rx_1(1 - 2x_0)$$

Substituting the value of x_1:

$$D_t^\alpha x_2(t) = r\frac{t^\alpha}{\Gamma(\alpha+1)}[rx_0(1 - x_0)](1 - 2x_0)$$
$$D_t^\alpha x_2(t) = \frac{t^\alpha r^2}{\Gamma(\alpha+1)}[x_0(1 - x_0)(1 - 2x_0)]$$

Applying the fractional integration operator J_t^α; we have:

$$x_2(t) = J_t^\alpha\left\{\frac{t^\alpha r^2}{\Gamma(\alpha+1)}[x_0(1 - x_0)(1 - 2x_0)]\right\}$$
$$= \frac{t^\alpha}{\Gamma(\alpha+1)}\left\{\frac{t^\alpha r^2}{\Gamma(\alpha+1)}[x_0(1 - x_0)(1 - 2x_0)]\right\}$$
$$= \frac{r^2 t^{2\alpha}}{\Gamma(2\alpha+1)}[x_0(1 - x_0)(1 - 2x_0)]$$

Order p^3:

$$D_t^\alpha x_3(t) = r[x_2(1 - 2x_0) - {x_1}^2]$$

Substituting the values of x_1 and x_2, we have:

$$D_t^\alpha x_3(t) = r\left[\frac{r^2 t^{2\alpha}}{\Gamma(2\alpha+1)}[x_0(1 - x_0)(1 - 2x_0)](1 - 2x_0) - \left(\frac{t^\alpha}{\Gamma(\alpha+1)}[rx_0(1 - x_0)]\right)^2\right]$$

$$= \frac{x_0 r^3 t^{2\alpha}}{\Gamma(2\alpha+1)}[(1 - x_0)(1 - 2x_0)^2] - \frac{{x_0}^2 r^3 t^{2\alpha}}{\Gamma(2\alpha+1)}(1 - x_0)^2$$

$$= \frac{x_0 r^3 t^{2\alpha}}{\Gamma(2\alpha+1)}(1 - x_0)[(1 - 2x_0)^2 - x_0(1 - x_0)]$$

Applying the fractional integration operator J_t^α; we have:

$$x_3(t) = J_t^\alpha\left\{\frac{x_0 r^3 t^{2\alpha}}{\Gamma(2\alpha+1)}(1 - x_0)[(1 - 2x_0)^2 - x_0(1 - x_0)]\right\}$$

$$= \frac{t^\alpha}{\Gamma(\alpha+1)}\left\{\frac{x_0 r^3 t^{2\alpha}}{\Gamma(2\alpha+1)}(1 - x_0)[(1 - 2x_0)^2 - x_0(1 - x_0)]\right\}$$

$$= \frac{x_0 r^3 t^{3\alpha}(1-x_0)}{\Gamma(3\alpha+1)}[(1 - 2x_0)^2 - x_0(1 - x_0)]$$

The approximate solution to the Fractional logistic equation is assumed to be

$$x(t) = x_0 + px_1(t) + p^2 x_2(t) + p^3 x_3(t) + \cdots$$

Setting $p = 1$, the approximate solution becomes:

$$x(t) = x_0 + x_1(t) + x_2(t) + x_3(t) + \cdots$$

Substituting the values gives the approximate solution as:

$$x(t) = x_0 + \frac{t^\alpha}{\Gamma(\alpha+1)}[rx_0(1 - x_0)] + \frac{r^2 t^{2\alpha}}{\Gamma(2\alpha+1)}[x_0(1 - x_0)(1 - 2x_0)] + \frac{x_0 r^3 t^{3\alpha}(1-x_0)}{\Gamma(3\alpha+1)}[(1 - 2x_0)^2 - x_0(1 - x_0)] + \cdots$$

The solution obtained via the Homotopy Perturbation Method is expressed as a rapidly convergent series involving fractional powers of time. The appearance of the Gamma function in the solution arises naturally from the fractional integration process and reflects the nonlocal nature of the fractional derivative. Unlike classical derivatives, which depend only on the current state of the system, fractional derivatives incorporate the entire history of the system. Consequently, the Gamma function plays a crucial role in scaling the time-dependent terms in a way that preserves this history-dependent behavior. Furthermore, the method avoids discretization since it does not rely on step-by-step numerical approximations or finite difference schemes. Instead, HPM provides a continuous analytical approximation in the form of a series solution. This allows for a more accurate representation of the system dynamics without introducing numerical errors associated with discretization. The resulting solution also captures the memory effect inherent in fractional systems. The presence of fractional powers of time indicates that the system evolves more gradually compared to classical integer-order models, reflecting the influence of past states on current behavior. This feature makes the method particularly suitable for modeling real-world processes where memory and hereditary properties play a significant role.

IV. NUMERICAL SIMULATIONS

In this section, the simulations were performed over appropriate time intervals to ensure clear visualization of the solution behavior. A sufficient number of discretization points were used in MATLAB to produce smooth and accurate curves. The selected parameter values were chosen to highlight key characteristics of each model, including nonlinearity, diffusion effects, and fractional dynamics. The numerical results not only validate the analytical approximations obtained via the Homotopy Perturbation Method but also provide visual insight into the convergence properties and stability of the solutions. These simulations further demonstrate the capability of the method to capture essential physical and dynamical features of both integer-order and fractional-order systems.

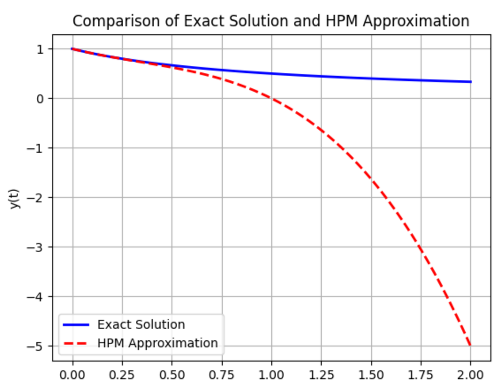

Figure 1: Comparison of Exact Solution and HPM Approximation for the non-linear ODE

Figure 1 shows that the HPM approximation obtained using a finite number of terms, closely follows the exact solution for small values of t, demonstrating the rapid convergence and high accuracy of the method. However, deviations increase as t grows, indicating that higher-order terms are required for improved accuracy over larger intervals.

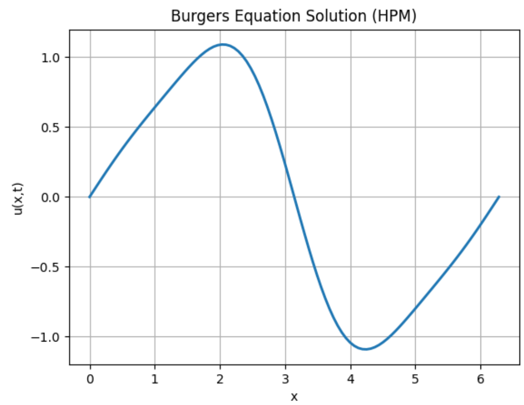

Figure 2: HPM approximate solution for the Burgers' equation

Figure 2 shows the approximate solution of Burgers' equation obtained using the Homotopy Perturbation Method at a fixed time $t = 0.5$. The solution captures both the nonlinear convective effects and the diffusive behavior governed by the viscosity parameter, illustrating the effectiveness of HPM in handling nonlinear partial differential equations.

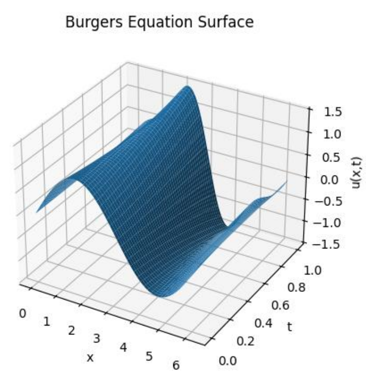

Figure 3: Surface plot of Burgers' equation solution (HPM)

Figure 3 shows the surface plot of the approximate solution of Burgers' equation obtained via the Homotopy Perturbation Method. The figure illustrates the evolution of the solution $u(x, t)$over space and time, highlighting the interplay between nonlinear convection and diffusion.

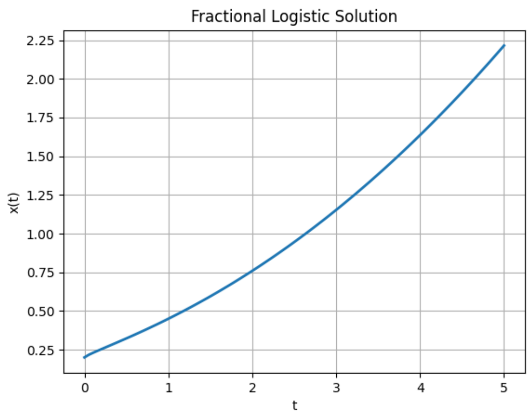

Figure 4: Solution of the Fractional Logistic equation

Figure 4 shows the approximate solution of the fractional logistic equation obtained using the Homotopy Perturbation Method for fractional order $\alpha = 0.8$. The solution demonstrates the influence of fractional dynamics on system evolution, characterized by slower growth compared to the classical integer-order case.

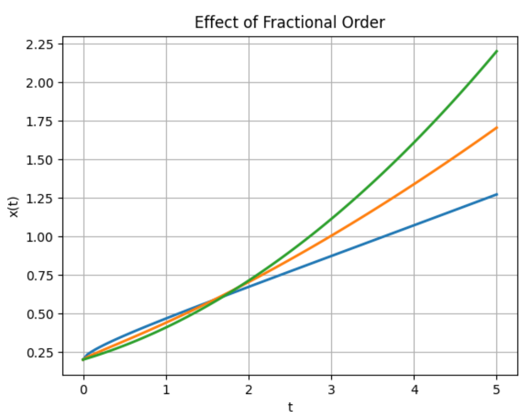

Figure 5: Effect of Fractional Order on Logistic Growth

Figure 5 shows comparison of approximate solutions of the fractional logistic equation for different values of the fractional order α. The results show that decreasing the fractional order leads to slower system growth, reflecting the memory effect inherent in fractional differential equations. The α values used are $\alpha = 1.0,\ \alpha = 0.8$, and $\alpha = 0.6$.

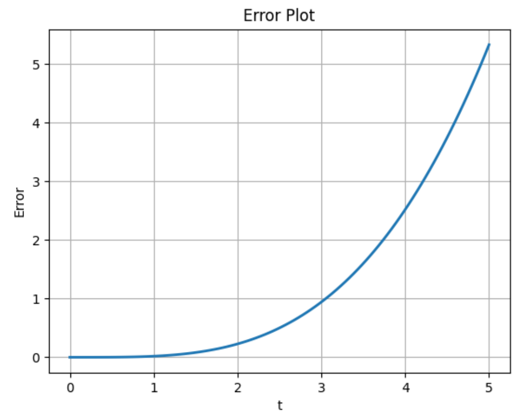

Figure 6: Error between Exact Solution and HPM Approximation

Figure 6 shows the absolute error between the exact solution and the Homotopy Perturbation Method approximation for the nonlinear ordinary differential equation. The small magnitude of the error confirms the high accuracy and rapid convergence of the HPM series solution.

V. RESULTS AND DISCUSSION

This section presents the results obtained from the application of the Homotopy Perturbation Method (HPM) to the selected nonlinear ordinary differential equation, Burgers' equation, and the fractional logistic equation. The accuracy, convergence behavior, and efficiency of the method are analyzed alongside graphical illustrations obtained from numerical simulations.

5.1 Convergence Behavior of HPM Solutions

The solutions obtained for all considered equations are expressed as rapidly convergent series. For the nonlinear ordinary differential equation, the HPM approximation closely matches the exact analytical solution, as illustrated in Figure 1. The graphical comparison shows that even a low-order approximation provides excellent agreement for small values of t, confirming the rapid convergence of the method. However, slight deviations observed for larger values of t indicate the need for higher-order terms to maintain accuracy over extended intervals.

For Burgers' equation, the approximate solution obtained using a few iterations effectively captures the essential dynamics of the system. As shown in Figure 2, the solution reflects both nonlinear convection and diffusion effects at a fixed time $t=0.5$. Furthermore, the surface plot in Figure 3 provides a comprehensive visualization of the solution evolution over space and time, clearly demonstrating the interaction between nonlinear steepening and viscous smoothing. These graphical results confirm that HPM converges efficiently while preserving the physical characteristics of the system and therefore is well-suited for non-linear partial differential equations.

In the case of the fractional logistic equation, the solution is expressed in terms of fractional powers of time and Gamma functions. The graphical representation in Figure 4 shows that the system evolves more gradually compared to classical models, reflecting the influence of fractional dynamics. The convergence of the HPM series remains satisfactory, although slightly slower than in integer-order cases due to the nonlocal nature of fractional derivatives.

5.2 Accuracy and Validation

The accuracy of the HPM solutions is validated through both analytical comparison and numerical analysis. In Figure 1, the close overlap between the exact solution and the HPM approximation confirms the high level of accuracy achieved with only a few terms in the series. This is further supported by the error analysis presented in Figure 6, where the absolute error remains very small across the domain, demonstrating the reliability of the method. For Burgers' equation, although an exact solution is not explicitly compared, the numerical results in Figures 2 and 3 exhibit physically consistent behavior. The solution demonstrates the expected smoothing effect due to viscosity and non-linear wave propagation, which aligns with known theoretical and numerical results. In the fractional logistic model, the absence of a simple closed-form exact solution necessitates comparison with numerical approximations. The results shown in Figure 4 are consistent with established behavior of fractional systems, confirming that HPM produces realistic and accurate approximations. A key advantage of the fractional model is its ability to capture memory effects, which is clearly illustrated in Figure 5. The comparison of solutions for different values of the fractional order $\alpha=1.0, 0.8,$and 0.6 shows that decreasing α leads to slower system growth. This behavior reflects the increasing influence of memory and hereditary properties, which are absent in classical integer-order models.

5.3 Comparison with Other Methods

The performance of HPM is compared with other semi-analytical methods such as the Adomian Decomposition Method (ADM) and the Variational Iteration Method (VIM). HPM demonstrates several advantages:

- It does not require the computation of Adomian polynomials, which simplifies implementation compared to ADM [11].
- It avoids the construction of correction functionals required in VIM, reducing computational complexity.
- It provides rapidly convergent series solutions with fewer iterations.

In contrast, ADM and VIM may involve more elaborate computations, especially for strongly nonlinear or fractional problems. These observations are consistent with previous comparative studies [5,10].

5.4 Interpretation of Results

The results highlight the robustness and versatility of HPM in solving both integer-order and fractional differential equations. For classical problems, the method produces solutions that closely approximate exact results, as demonstrated in Figure 1. For nonlinear partial differential equations such as Burgers' equation, the method successfully captures complex physical behaviors, as seen in Figures 2 and 3. In the fractional case, the presence of fractional powers of time and Gamma functions in the solution reflects the inherent memory effect of the system.

This is further supported by the graphical results in Figures 4 and 5, where the system exhibits slower dynamics as the fractional order decreases. Such behavior is characteristic of systems with hereditary properties and has significant implications for modeling real-world processes, including biological and epidemiological systems. On the whole, the combination of analytical solutions and numerical simulations demonstrates that HPM is a powerful and efficient tool for solving a wide range of nonlinear and fractional differential equations.

5.5 Conclusion

In this study, the Homotopy Perturbation Method (HPM) has been successfully applied to selected nonlinear and fractional differential equations, including an ordinary differential equation, Burgers' equation, and a fractional logistic model. The method produces accurate and rapidly convergent series solutions without requiring discretization, linearization, or small perturbation parameters. The effectiveness of the method is clearly supported by the numerical simulations. The close agreement between the exact and approximate solutions in Figure 1, along with the minimal error observed in Figure 6, confirms the high accuracy and rapid convergence of HPM. The graphical results for Burgers' equation in Figures 2 and 3 further demonstrate the method's ability to capture complex nonlinear and diffusive behaviors. In the fractional logistic model, the simulations in Figures 4 and 5 highlight the significant impact of fractional order on system dynamics. The observed slower growth for lower values of α confirms the presence of memory effects, illustrating the importance of fractional modeling in representing real-world systems more accurately. Comparative analysis indicates that HPM offers notable advantages over other semi-analytical methods, including simplicity, reduced computational effort, and ease of implementation. Its ability to handle strong nonlinearities and fractional operators makes it a valuable tool in applied mathematics, physics, and engineering. Future work may extend the application of HPM to more complex systems, including coupled fractional models, stochastic processes, and real-world applications in epidemiology and engineering. Additionally, integrating HPM with numerical techniques may further enhance its applicability and accuracy. In conclusion, the Homotopy Perturbation Method remains a robust, efficient, and versatile approach for solving nonlinear and fractional differential equations, as confirmed by both analytical results and numerical simulations.

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