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The Complex Form of A New Generalization of The Bernstein Operator, Depending on A Non-Negative Real Parameter

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Abstract - In this chapter we discuss the complex form of a new generalization of the Bernstein operator, depending on a non-negative real parameter. In which we obtained quantitative upper estimate for simultaneous approximation. In this chapter, we discussed the complex form of a new generalization of the Bernstein operator, depending on a non-negative real parameter and obtained quantitative upper estimate for simultaneous approximation, a qualitative Voronovskaja type result and the exact order of approximation. Also, we present some shape preserving properties of the complex _-Bernstein operator such as univalence, starlikeness, convexity and spirallikeness. We obtained quantitative upper estimate for simultaneous approximation, a qualitative Voronovskaja type result and the exact order of approximation.

I. INTRODUCTION

The classical Bernstein operators given by

$$B_m(h;y) = \sum_{t=0}^{m} h\left(\frac{t}{m}\right)\binom{m}{t} y^t (1-y)^{m-t}, \quad y \in [0,1]. \qquad (1)$$

for any $m \in N$ and h $\in \mathcal{C}[0,1]$, the space of all real valued continuous functions on $[0,1]$, were proposed by Bernstein as one of the simplest way to prove Weierstrass approximation theorem.

Bernstein operator have many advantages in terms of their elegant structure, simplicity and useful approximation properties. Chen, X., Tan, J., Liu, Z., Xie and J., [10] constructed a new family of generalized Bernstein operators which is called as _-Bernstein operator, depending on a non-negative real parameter, is given by

$$\tau_{m,\alpha}(h;y) = \sum_{t=0}^{m} h\left(\frac{t}{m}\right) q^{(\alpha)}_{m,t}(y), \qquad (2)$$

for any function $h(y)$ defined on $[0,1]$, each positive integer m and any fixed real α. Here, for $t = 0, 1, \cdots, m$, the α-Bernstein polynomial $q^{(\alpha)}_{m,t}(y)$ of degree m is defined by $q^{(\alpha)}_{1,0}(y) = 1 - y$, $q^{(\alpha)}_{1,1}(y) = y$ and

$$q^{(\alpha)}_{m,t}(y) = \left[\binom{m-2}{t}(1-\alpha)(y) + \binom{m-2}{t-2}(1-\alpha)(1-y) + \binom{m}{t}\alpha y(1-y)\right] \times y^{t-1}(1-y)^{m-t-1}.$$

where $m \geq 2$, $y \in [0,1]$ and the binomial coefficients $\binom{m}{t}$ are given by

$$\binom{m}{t} = \begin{cases} \dfrac{m!}{(m-t)!t!}, & \text{if } 0 \leq t \leq m, \\ 0, & else. \end{cases}$$

It is obvious that for α = 1, the α-Bernstein operator becomes the classical Bernstein polynomial. Moreover, the α-Bernstein operators are linear positive operators for $0 \leq \alpha \leq 1$. In [10], the authors gave some elementary properties and proved the uniform convergency of the sequence of the α -Bernstein operators to $\zeta \in C[0;\ 1]$ with the help of the well known Korovkin theorem. They obtained the rate of convergence and Voronovskaja type theorem for the α-Bernstein operator. Also, they estimated an upper bound for the approximation error by means of the modulus of continuity and proved that the α-Bernstein operator satisfies some shape preserving results. The problem of approximation of complex operators has attracted attention of many researchers. Some approximation properties of complex

Bernstein polynomials in various domains in complex plane were obtained without any quantitative estimate. In [14], Gal obtained quantitative estimates for the convergence and Voronovskaja's theorem in addition to the results obtained in [22]. Also, Gal compiled the over convergence properties of the well known complex operators [14].

Motivated from the real case in [10] and the above works on several operators in complex domain, we consider complex form of the new generalized Bernstein operators defined as follows

$$\tau_{m,\alpha}(h;z) = \sum_{t=0}^{m} h_t q_{m,t}^{(\alpha)}(z), \qquad (3)$$

where $h_t = h\left(\frac{t}{m}\right)$, m is a positive integer, α is a real parameter satisfying the condition $0 \le \alpha \le 1$, $z \in \mathbb{C}$ and h is a complex-valued analytic function in an open disk centered at the origin with $R_1 > 1$. For $t = 0, 1, \cdots, m$ the α-Bernstein polynomial $q_{m,t}^{(\alpha)}(z)$ of degree m is defined by $q_{1,0}^{(\alpha)}(z) = 1 - z$, $q_{1,1}^{(\alpha)}(z) = z$ and for $m \ge 2$

$$q_{m,t}^{(\alpha)}(z) = \left[\binom{m-2}{t}(1-\alpha)(z) + \binom{m-2}{t-2}(1-\alpha)(1-z) + \binom{m}{t}\alpha z(1-z)\right] \times z^{t-1}(1-z)^{m-t-1}.$$

Here, firstly we obtain quantitative upper estimate for the complex α-Bernstein operator and its derivatives on compact disks. Then, we obtain the qualitative Voronovskaja type result and the exact order of approximation for these operators. Finally, we prove that the complex α-Bernstein operators attached to an analytic function preserve the univalence, starlikeness, convexity and
spirallikeness in the unit disk.

Approximation by Complex α-Bernstein Operator

Firstly we will give the following results which include some properties of the complex α −Bernstein operator. We use the denotation $lt(z) := z^t$; $t \in N \cup \{0\}$; $z \in \mathbb{C}$ and

$$D_{R_1} = \{z \in \mathbb{C} : |z| < R_1,\ 1 < R_1 < \infty\}.$$

Let Δ_t^b denote the *finite difference of order t with step* b, that is

$$\Delta_b^t h(y) = t! b^t [y, y+b, \cdots, y+tb; h]. \qquad (4)$$

The other representations of the _-Bernstein operator were established by Chen, X., Tan, J., Liu, Z., Xie and J., for functions of real variable [10]. This formulas hold in complex setting too.

Theorem 2.1*. The complex _-Bernstein operator defined by (3) has the following another representation*

$$\tau_{m,\alpha}(h;z) = (1-\alpha)\sum_{t=0}^{m-1} \Gamma_t \binom{m}{t} z^t (1-z)^{m-t-1} + \alpha \sum_{t=0}^{m} h_t \binom{m}{t} z^t (1-z)^{m-t}, \quad (5)$$

Where

$$\Gamma_t = \left(1 - \frac{t}{m-1}\right) h_t + \frac{t}{m-1} h_{t+1}, \quad m \ge 2.$$

Proof. We know that

$$\tau_{m,\alpha}(h;z) = \sum_{t=0}^{m} h_t q_{m,t}^{(\alpha)}(z),$$

Where

$$q_{m,t}^{(\alpha)}(z) = \left[\binom{m-2}{t}(1-\alpha)(z) + \binom{m-2}{t-2}(1-\alpha)(1-z) + \binom{m}{t}\alpha z(1-z)\right] \times z^{t-1}(1-z)^{m-t-1}.$$

So, we have

$$\tau_{m,\alpha}(h;z) = \sum_{t=0}^{m} h_t \left[\binom{m-2}{t}(1-\alpha)(z) + \binom{m-2}{t-2}(1-\alpha)(1-z) + \binom{m}{t}\alpha z(1-z)\right] \times z^{t-1}(1-z)^{m-t-1}$$

$$\tau_{m,\alpha}(h;z) = \sum_{t=0}^{m} h_t \binom{m-2}{t}(1-\alpha)z^t(1-z)^{m-t-1} + \sum_{t=0}^{m} h_t \binom{m-2}{t-2}(1-\alpha)z^{t-1}(1-z)^{m-t} + \sum_{t=0}^{m} h_t \binom{m}{t}\alpha z^t(1-z)^{m-t}$$

$$\tau_{m,\alpha}(h;z) = (1-\alpha)\sum_{t=0}^{m} h_t \binom{m-2}{t} z^t(1-z)^{m-t-1} + \sum_{t=0}^{m} h_t \binom{m-2}{t-2} z^{t-1}(1-z)^{m-t} + \sum_{t=0}^{m} h_t \binom{m}{t}\alpha z^t(1-z)^{m-t}$$

We can write the above equation as

$$\tau_{m,\alpha}(h;z) = (1-\alpha)(b_1+b_2) + \sum_{t=0}^{m} h_t \binom{m}{t}\alpha z^t(1-z)^{m-t} \quad (6)$$

Where

$$b_1 = \sum_{t=0}^{m} h_t \binom{m-2}{t} z^t(1-z)^{m-t-1} \quad \text{and} \quad b_2 = \sum_{t=0}^{m} h_t \binom{m-2}{t-2} z^{t-1}(1-z)^{m-t}$$

Note that the terms corresponding to $t = m$ in b_1 and $t = 0$ in b_2 are both zero. The b_1 and b_2 can respectively be expressed as

$$b_1 = \sum_{t=0}^{m-1} h_t \binom{m-2}{t} z^t(1-z)^{m-t-1} \quad \text{and} \quad b_2 = \sum_{t=1}^{m} h_t \binom{m-2}{t-2} z^{t-1}(1-z)^{m-t}$$

where $\binom{m-2}{m-1} = 0$ in b_1 and $\binom{m-2}{-1} = 0$ in b_2. Replacing t by $t+1$ in the summation of b_2, we obtain

$$b_2 = \sum_{t=0}^{m-1} h_{t+1} \binom{m-2}{t-2} z^{t-1}(1-z)^{m-t}$$

So, it follows that

$$b_1 + b_2 = \sum_{t=0}^{m-1} \left[\binom{m-2}{t} h_t + \binom{m-2}{t-1} h_{t+1}\right] z^t(1-z)^{m-t-1} \quad (7)$$

Since

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$$\begin{aligned}
\binom{m-2}{t} &= \frac{(m-2)!}{t!(m-2-t)!} \\
\binom{m-2}{t} &= \frac{(m-2)!(m-1-t)}{t!(m-1-t)!} \\
\binom{m-2}{t} &= \frac{(m-2)!(m-1)-t(m-2)!}{t!(m-1-t)!} \\
\binom{m-2}{t} &= \frac{(m-1)!-t(m-2)!}{(m-1-t)!} \\
\binom{m-2}{t} &= \frac{(m-1)!}{t!(m-1-t)} - \frac{t(m-1)(m-2)!}{t!(m-1-t)!(m-1)} \\
\binom{m-2}{t} &= \binom{m-1}{t} - \frac{t}{m-1}\frac{(m-1)!}{t!(m-1-t)!} \\
\binom{m-2}{t} &= \left(1-\frac{t}{m-1}\right)\binom{m-1}{t} \qquad (8)
\end{aligned}$$

And

$$\begin{aligned}
\binom{m-2}{t-1} &= \frac{(m-2)!}{(t-1)!(m-2-t+1)!} \\
\binom{m-2}{t-1} &= \frac{(m-2)!}{(t-1)!(m-1-t)!} \\
\binom{m-2}{t-1} &= \frac{t}{m-1}\frac{(m-1)(m-2)!}{t(t-1)!(m-1-t)!} \\
\binom{m-2}{t-1} &= \frac{t}{m-1}\frac{(m-1)!}{t!(m-1-t)!} \\
\binom{m-2}{t-1} &= \frac{t}{m-1}\binom{m-1}{t} \qquad (9)
\end{aligned}$$

Putting the values from (9) and (8) in equation (7), we get

$$b_1 + b_2 = \sum_{t=0}^{m-1}\left(1-\frac{t}{m-1}\right)h_t + \frac{t}{m-1}h_{t+1}\binom{m-1}{t}z^t(1-z)^{m-t-1} \qquad (10)$$

By equation (6) and (10), we can write

$$\begin{aligned}
\tau_{m,\alpha}(h;z) &= (1-\alpha)\sum_{t=0}^{m-1}\left[\left(1-\frac{t}{m-1}\right)h_t + \frac{t}{m-1}h_{t+1}\right]\binom{m-1}{t}z^t(1-z)^{n-t-1} \\
&\quad +\alpha\sum_{t=0}^{m}h_t\binom{m}{t}z^t(1-z)^{m-t} \\
\tau_{m,\alpha}(h;z) &= (1-\alpha)\sum_{t=0}^{m-1}\Gamma_t\binom{m-1}{t}z^t(1-z)^{m-t-1} + \alpha\sum_{t=0}^{m}h_t\binom{m}{t}z^t(1-z)^{m-t}
\end{aligned}$$

Where

$$\Gamma_t = \left(1 - \frac{t}{m-1}\right) h_t + \frac{t}{m-1} h_{t+1}$$

Theorem 2.2. *The complex α-Bernstein operator can be expressed by means of finite differences as follows*

$$\tau_{m,\alpha}(h;z) = \sum_{q=0}^{m} \left[(1-\alpha)\binom{m-1}{q} \left\{ \Delta^q_{1/m} h(0) + \frac{q}{m-1} \Delta^q_{1/m} h\left(\frac{1}{m}\right) \right\} + \alpha \binom{m}{q} \Delta^q_{1/m} h(0) \right] z^q, \quad (11)$$

where $\Delta^q_{\frac{1}{m}}$ is given by (4) with $b = \frac{1}{m}$.

Proof:

Since

$$h_t = h\left(\frac{t}{m}\right), \qquad \Gamma_t = \left(1 - \frac{t}{m-1}\right) h_t + \frac{t}{m-1} h_{t+1}$$

So,

$$\Gamma_0 = h_0$$

where $\Delta^q_{\frac{1}{m}}$ is given by

$$\Delta^q_{1/m} h(z) = \frac{q!}{m^q} \left[z, z + \frac{1}{m}, \cdots, z + \frac{q}{m}; h \right]$$

[10] Expanding the term $(1 - z)^{m-t-1}$ of (7), we have

$$b_1 + b_2 = \sum_{t=0}^{m-1} \Gamma_t \binom{m-1}{t} z^t \sum_{j=0}^{m-t-1} (-1)^s \binom{m-t-1}{t} z^s$$

Let us put $q = t + s$, we can write

$$\sum_{t=0}^{m-1} \sum_{s=0}^{m-t-1} = \sum_{q=0}^{m-1} \sum_{t=0}^{q}$$

We also have

$$\binom{m-1}{t}\binom{m-t-1}{s} = \frac{(m-1)!}{(m-1-t)!t!} \frac{(m-t-1)!}{(m-t-1-s)!s!}$$
$$\binom{m-1}{t}\binom{m-t-1}{s} = \frac{(m-1)!}{t!(m-t-1-s)!s!}$$

Put $s = q - t$ in above equation

$$\binom{m-1}{t}\binom{m-t-1}{s} = \frac{(m-1)!}{t!(m-t-1-q+t)!(q-t)!}$$
$$\binom{m-1}{t}\binom{m-t-1}{s} = \frac{(m-1)!}{t!(m-q-1)!(q-t)!}$$
$$\binom{m-1}{t}\binom{m-t-1}{s} = \frac{q!}{(q-t)!t!} \frac{(m-1)!}{(m-q-1)!q!}$$
$$\binom{m-1}{t}\binom{m-t-1}{s} = \binom{q}{t}\binom{m-1}{q}$$

So we can write the double summation as

$$b_1 + b_2 = \sum_{q=0}^{m-1} \binom{m-1}{t} z^q \sum_{s=0}^{m-t-1} (-1)^{q-t} \binom{q}{t} \Gamma_t$$

It follows from using the expansion for a higher-order forward difference that

$$b_1 + b_2 = \sum_{q=0}^{m} \binom{m-1}{t} z^q \Delta^q \Gamma_0$$

$$b_1 + b_2 = \sum_{q=0}^{m} \binom{m-1}{t} z^q \Delta^q \Gamma_0$$

The term corresponding to $q = m$ in the above sum is zero. Similarly we can prove that

$$\sum_{t=0}^{m} h_t \binom{m}{t} z^t (1-z)^{m-t} = \sum_{q=0}^{m} \binom{m}{q} z^q \Delta^q h_0$$

Therefore, we have

$$\tau_{m,\alpha}(h; z) = \sum_{q=0}^{m} \left[(1-\alpha) \binom{m-1}{q} \Delta^q \Gamma_0 + \alpha \binom{m}{q} \Delta^q h_0 \right] z^q$$

This completes the proof.

Theorem 2.3. *For all $a \in N \cup \{0\}, n \in N, \alpha \epsilon [0;\ 1]$ and $z \epsilon \mathbb{C}$, we have*

$$\begin{aligned} \tau_{m,\alpha}(l_{a+1}; z) &= \frac{z(1-z)}{m} (\tau_{m,\alpha}(l_a; z))' + \left[z + \frac{1-z}{m} \right] \tau_{m,\alpha}(l_a; z) \\ &+ \frac{1-\alpha}{m} \left(\frac{m-1}{m} \right)^a [B_{m-1}(l_{a+1}; z) - B_{m-1}(l_a; z)] \\ &- \frac{\alpha(1-z)}{m} B_m(l_a; z), \end{aligned} \quad (12)$$

where B_m is the mth complex Bernstein operator.

Proof. From (5) we can write

$$\begin{aligned} \tau_{m,\alpha}(l_a; z) &= (1-\alpha) \sum_{t=0}^{m-1} \binom{m-1}{t} z^t (1-z)^{m-t-1} \left[\left(1 - \frac{t}{m-1}\right) \left(\frac{t}{m}\right)^a + \frac{t}{m-1} \left(\frac{t+1}{m}\right)^a \right] \\ &+ \alpha \sum_{t=0}^{m} \binom{m}{t} z^t (1-z)^{m-t} \left(\frac{t}{m}\right)^a \end{aligned}$$

Differentiating $\tau_{m,\alpha}(lt;\ z)$ with respect to $z \neq 0$, by some calculations we get

$$\begin{aligned} (\tau_{m,\alpha}(b_a; z))' &= (1-\alpha) \sum_{t=0}^{m-1} \binom{m-1}{t} [t z^{t-1} (1-z)^{m-t-1} - (m-t-1) z^t (1-z)^{m-t-2}] \\ &\times \left[\left(1 - \frac{t}{m-1}\right) \left(\frac{t}{m}\right)^a + \frac{t}{m-1} \left(\frac{t+1}{m}\right)^a \right] \\ &+ \alpha \sum_{t=0}^{m} \binom{m}{t} [t z^{t-1} (1-z)^{m-t} - (m-t) z^t (1-z)^{m-t-1}] \left(\frac{t}{m}\right)^a \end{aligned}$$

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$$\begin{aligned}
(\tau_{m,\alpha}(b_a;z))' &= (1-\alpha)\sum_{t=0}^{m-1}\binom{m-1}{t}\left[tz^{t-1}(1-z)^{m-t-1}\right.\\
&\left.-(m-1)z^t(1-z)^{m-t-2}+tz^t(1-z)^{m-t-2}\right]\\
&\times\left[\left(1-\frac{t}{m-1}\right)\left(\frac{t}{m}\right)^a+\frac{t}{m-1}\left(\frac{t+1}{m}\right)^a\right]\\
&+\alpha\sum_{t=0}^{m}\binom{m}{t}\left[tz^{t-1}(1-z)^{m-t}-mz^t(1-z)^{m-t-1}+tz^t(1-z)^{m-t-1}\right]\left(\frac{t}{m}\right)^a\\
(\tau_{m,\alpha}(b_a;z))' &= (1-\alpha)\sum_{t=0}^{m-1}\binom{m-1}{t}\left[tz^{t-1}(1-z)^{m-t-2}\left(1-z+z\right)-(m-1)z^t(1-z)^{m-t-2}\right]\\
&\times\left[\left(1-\frac{t}{m-1}\right)\left(\frac{t}{m}\right)^a+\frac{t}{m-1}\left(\frac{t+1}{m}\right)^a\right]\\
&+\alpha\sum_{t=0}^{m}\binom{m}{t}\left[tz^{t-1}(1-z)^{m-t-1}\left(1-z+z\right)-mz^t(1-z)^{m-t-1}\right]\left(\frac{t}{m}\right)^a\\
(\tau_{m,\alpha}(b_a;z))' &= (1-\alpha)\sum_{t=0}^{m-1}\binom{m-1}{t}\left[tz^{t-1}(1-z)^{m-t-2}-(m-1)z^t(1-z)^{m-t-2}\right]\\
&\times\left[\left(1-\frac{t}{m-1}\right)\left(\frac{t}{m}\right)^a+\frac{t}{m-1}\left(\frac{t+1}{m}\right)^a\right]\\
&+\alpha\sum_{t=0}^{m}\binom{m}{t}\left[tz^{t-1}(1-z)^{m-t-1}-mz^t(1-z)^{m-t-1}\right]\left(\frac{t}{m}\right)^a\\
(\tau_{m,\alpha}(b_a;z))' &= (1-\alpha)\sum_{t=0}^{m-1}\binom{m-1}{t}\left[tz^{t-1}(1-z)^{m-t-2}\frac{z(1-z)}{z(1-z)}-(m-1)z^t(1-z)^{m-t-2}\right.\\
&\left.\times\frac{(1-z)}{(1-z)}\right]\left[\left(1-\frac{t}{m-1}\right)\left(\frac{t}{m}\right)^a+\frac{t}{m-1}\left(\frac{t+1}{m}\right)^a\right]\\
&+\alpha\sum_{t=0}^{m}\binom{m}{t}\left[tz^{t-1}(1-z)^{m-t-1}\frac{z(1-z)}{z(1-z)}-mz^t(1-z)^{m-t-1}\frac{(1-z)}{(1-z)}\right]\left(\frac{t}{m}\right)^a\\
(\tau_{m,\alpha}(b_a;z))' &= \frac{1-\alpha}{z(1-z)}\sum_{t=0}^{m-1}\binom{m-1}{t}z^t(1-z)^{m-t-1}t\\
&\times\left[\left(1-\frac{t}{m-1}\right)\left(\frac{t}{m}\right)^a+\frac{t}{m-1}\left(\frac{t+1}{m}\right)^a\right]\\
&-\frac{(m-1)(1-\alpha)}{1-z}\sum_{t=0}^{m-1}\binom{m-1}{t}z^t(1-z)^{m-t-1}\\
&\times\left[\left(1-\frac{t}{m-1}\right)\left(\frac{t}{m}\right)^a+\frac{t}{m-1}\left(\frac{t+1}{m}\right)^a\right]\\
&+\frac{\alpha}{z(1-z)}\sum_{t=0}^{m}\binom{m}{t}z^t(1-z)^{m-t}t\left(\frac{t}{m}\right)^a\\
&-\frac{\alpha m}{(1-z)}\sum_{t=0}^{m}\binom{m}{t}z^t(1-z)^{m-t}\left(\frac{t}{m}\right)^a
\end{aligned}$$

$$\begin{aligned}(\tau_{m,\alpha}(b_a;z))' &= \frac{1-\alpha}{z(1-z)}\sum_{t=0}^{m-1}\binom{m-1}{t}z^t(1-z)^{m-t-1}t\left[\left(1-\frac{t}{m-1}\right)\left(\frac{t}{m}\right)^a\right]\\ &+\frac{1-\alpha}{z(1-z)}\sum_{t=0}^{m-1}\binom{m-1}{t}z^t(1-z)^{m-t-1}t\left[\frac{t}{m-1}\left(\frac{t+1}{m}\right)^a\right]\\ &-\frac{(m-1)(1-\alpha)}{1-z}\sum_{t=0}^{m-1}\binom{m-1}{t}z^t(1-z)^{m-t-1}\\ &\times\left[\left(1-\frac{t}{m-1}\right)\left(\frac{t}{m}\right)^a+\frac{t}{m-1}\left(\frac{t+1}{m}\right)^a\right]\\ &+\frac{\alpha}{z(1-z)}\sum_{t=0}^{m}\binom{m}{t}z^t(1-z)^{m-t}t\left(\frac{t}{m}\right)^a\\ &-\frac{\alpha m}{(1-z)}\sum_{t=0}^{m}\binom{m}{t}z^t(1-z)^{m-t}\left(\frac{t}{m}\right)^a\end{aligned}$$

$$\begin{aligned}(\tau_{m,\alpha}(b_a;z))' &= \frac{m(1-\alpha)}{z(1-z)}\sum_{t=0}^{m-1}\binom{m-1}{t}z^t(1-z)^{m-t-1}\left[\left(1-\frac{t}{m-1}\right)\left(\frac{t}{m}\right)^{a+1}\right]\\ &+\frac{m(1-\alpha)}{z(1-z)}\sum_{t=0}^{m-1}\binom{m-1}{t}z^t(1-z)^{m-t-1}\left[\frac{t}{m-1}\left(\frac{t+1}{m}\right)^{a+1}\right]\end{aligned}$$

$$\begin{aligned}&-\frac{1-\alpha}{z(1-z)}\sum_{t=0}^{m-1}\binom{m-1}{t}z^t(1-z)^{m-t-1}\left[\frac{t}{m-1}\left(\frac{t+1}{m}\right)^a\right]\\ &-\frac{(m-1)(1-\alpha)}{1-z}\sum_{t=0}^{m-1}\binom{m-1}{t}z^t(1-z)^{m-t-1}\\ &\times\left[\left(1-\frac{t}{m-1}\right)\left(\frac{t}{m}\right)^a+\frac{t}{m-1}\left(\frac{t+1}{m}\right)^a\right]\\ &+\frac{\alpha m}{z(1-z)}\sum_{t=0}^{m}\binom{m}{t}z^t(1-z)^{m-t}\left(\frac{t}{m}\right)^{a+1}\\ &-\frac{\alpha m}{(1-z)}\sum_{t=0}^{m}\binom{m}{t}z^t(1-z)^{m-t}\left(\frac{t}{m}\right)^a\end{aligned}$$

$$
\begin{aligned}
(\tau_{m,\alpha}(b_a;z))' &= \frac{m(1-\alpha)}{z(1-z)}\sum_{t=0}^{m-1}\binom{m-1}{t}z^t(1-z)^{m-t-1} \\
&\times\left[\left(1-\frac{t}{m-1}\right)\left(\frac{t}{m}\right)^{a+1}+\frac{t}{m-1}\left(\frac{t+1}{m}\right)^{a+1}\right] \\
&-\frac{1-\alpha}{z(1-z)}\sum_{t=0}^{m-1}\binom{m-1}{t}z^t(1-z)^{m-t-1}\left[\frac{t}{m-1}\left(\frac{t+1}{m}\right)^{a}\right] \\
&-\frac{(m-1)(1-\alpha)}{1-z}\sum_{t=0}^{m-1}\binom{m-1}{t}z^t(1-z)^{m-t-1} \\
&\times\left[\left(1-\frac{t}{m-1}\right)\left(\frac{t}{m}\right)^{a}+\frac{t}{m-1}\left(\frac{t+1}{m}\right)^{a}\right] \\
&+\frac{\alpha m}{z(1-z)}\sum_{t=0}^{m}\binom{m}{t}z^t(1-z)^{m-t}\left(\frac{t}{m}\right)^{a+1} \\
&-\frac{\alpha m}{(1-z)}\sum_{t=0}^{m}\binom{m}{t}z^t(1-z)^{m-t}\left(\frac{t}{m}\right)^{a} \\
(\tau_{m,\alpha}(b_a;z))' &= \frac{m}{z(1-z)}\tau_{m,\alpha}(l_{a+1};z)-\frac{1}{z(1-z)}\tau_{m,\alpha}(l_a;z) \\
&+\frac{1-\alpha}{z(1-z)}\sum_{t=0}^{m-1}\binom{m-1}{t}z^t(1-z)^{m-t-1}\left(\frac{t}{m}\right)^{a}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1-\alpha}{z(1-z)}\sum_{t=0}^{m-1}\binom{m-1}{t}z^t(1-z)^{m-t-1}\frac{t}{m-1}\left(\frac{t}{m}\right)^{a} \\
&+\frac{\alpha}{z(1-z)}\sum_{t=0}^{m}\binom{m}{t}z^t(1-z)^{m-t}\left(\frac{t}{m}\right)^{a} \\
&-\frac{m-1}{1-z}\tau_{m,\alpha}(l_a;z)+\frac{\alpha(m-1)}{(1-z)}\sum_{t=0}^{m}\binom{m}{t}z^t(1-z)^{m-t}\left(\frac{t}{m}\right)^{a} \\
&-\frac{\alpha m}{(1-z)}\sum_{t=0}^{m}\binom{m}{t}z^t(1-z)^{m-t}\left(\frac{t}{m}\right)^{a}
\end{aligned}
$$

$$
\begin{aligned}
(\tau_{m,\alpha}(b_a;z))' &= \frac{m}{z(1-z)}\tau_{m,\alpha}(l_{a+1};z)-\frac{[1+(m-1)z]}{z(1-z)}\tau_{m,\alpha}(l_a;z) \\
&+\frac{1-\alpha}{z(1-z)}\left(\frac{m-1}{m}\right)^{a}\sum_{t=0}^{m-1}\binom{m-1}{t}z^t(1-z)^{m-t-1}\left(\frac{t}{m-1}\right)^{a} \\
&-\frac{1-\alpha}{z(1-z)}\left(\frac{m-1}{m}\right)^{a}\sum_{t=0}^{m-1}\binom{m-1}{t}z^t(1-z)^{m-t-1}\left(\frac{t}{m-1}\right)^{a+1} \\
&+\frac{\alpha}{z}\sum_{t=0}^{m}\binom{m}{t}z^t(1-z)^{m-t}\left(\frac{t}{m}\right)^{a} \\
(\tau_{m,\alpha}(b_a;z))' &= \frac{m}{z(1-z)}\tau_{m,\alpha}(l_{a+1};z)-\frac{[1+(m-1)z]}{z(1-z)}\tau_{m,\alpha}(l_a;z) \\
&-\frac{1-\alpha}{z(1-z)}\left(\frac{m-1}{m}\right)^{a}[B_{m-1}(l_{a+1};z)-B_{m-1}(l_a;z)]+\frac{\alpha}{z}B_m(l_a;z)
\end{aligned}
$$

which gives the desired statement.

Lemma 2.1. *Let* r_1 *and* R_1 *be constants such that* $1 \leq r_1 \leq R_1$*: Also,* h *is analytic in* D_{R_1} *with* $h(z) = \sum_{a=0}^{\infty} d_a z^a$ *Then for all* $z \in D_{r_1} := \{z \in \mathbb{C} : |z| \leq r_1\}$*,* $m \in N$ *and* $\alpha \in [0;\ 1]$*, we have*

$$\tau_{m,\alpha}(h; z) = \sum_{a=0}^{\infty} d_a \tau_{m,\alpha}(l_a; z).$$

Proof. For any $u \in N$, we have $h_u(z) = \sum_{s=0}^{u} d_s z^s,\ |z| \leq r_1.$ Then since $\tau_{m,\alpha}$ is linear, we obviously obtain

$$\tau_{m,\alpha}(h_u; z) = \sum_{a=0}^{u} d_a \tau_{m,\alpha}(l_a; z)$$

for any fixed $m \in N$ and $|z| \leq r_1$. For all $|z| \geq r_1, r_1 \geq 1$, by (5) we have

$$\begin{aligned}
|\tau_{m,\alpha}(h_u; z) - \tau_{m,\alpha}(h; z)| &= \Bigg|(1-\alpha)\sum_{t=0}^{m-1}\binom{m-1}{t} z^t(1-z)^{m-t-1}\left[\left(1-\frac{t}{m-1}\right)h_u + \frac{t}{m-1}h_u\right] \\
&\quad + \alpha\sum_{t=0}^{m}\binom{m}{t} z^t(1-z)^{m-t}h_u - \Bigg[(1-\alpha)\sum_{t=0}^{m-1}\binom{m-1}{t} z^t(1-z)^{m-t-1} \\
&\quad \times\left[\left(1-\frac{t}{m-1}\right)h + \frac{m}{m-1}h\right] + \alpha\sum_{t=0}^{m}\binom{m}{t} z^t(1-z)^{m-t}h\Bigg]\Bigg| \\
|\tau_{m,\alpha}(h_u; z) - \tau_{m,\alpha}(h; z)| &= \Bigg|(1-\alpha)\sum_{t=0}^{m-1}\binom{m-1}{t} z^t(1-z)^{m-t-1} \\
&\quad \times\left[\left(1-\frac{t}{m-1}\right)(h_u-h) + \frac{t}{m-1}(h_u-h)\right] \\
&\quad + \alpha\sum_{t=0}^{m}\binom{m}{t} z^t(1-z)^{m-t}(h_u-h)\Bigg|
\end{aligned}$$

Applying Triangular Inequality to the above equation and we know that Triangular Inequality is

$$|a+b| \leq |a| + |b|$$

$$\begin{aligned}
|\tau_{m,\alpha}(h_u; z) - \tau_{m,\alpha}(h; z)| &\leq \Bigg|(1-\alpha)\sum_{t=0}^{m-1}\binom{m-1}{t} z^t(1-z)^{m-t-1} \\
&\quad \times\left[\left(1-\frac{t}{m-1}\right)(h_u-h) + \frac{t}{m-1}(h_u-h)\right]\Bigg| \\
&\quad + \left|\alpha\sum_{t=0}^{m}\binom{m}{t} z^t(1-z)^{m-t}(h_u-h)\right| \\
|\tau_{m,\alpha}(h_u; z) - \tau_{m,\alpha}(h; z)| &\leq (1-\alpha)\sum_{t=0}^{m-1}\binom{m-1}{t}\left|z^t(1-z)^{m-t-1}\right| \\
&\quad \times\left[\left(1-\frac{t}{m-1}\right)|h_u-h| + \frac{t}{m-1}|h_u-h|\right] \\
&\quad + \alpha\sum_{t=0}^{m}\binom{m}{t}\left|z^t(1-z)^{m-t}\right||h_u-h|
\end{aligned}$$

Now since

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$$\|P_m(z)\| = \max\{|P_m(z)|; \ \ |z| \leq r_1\}$$

So,

$$\begin{aligned} |\tau_{m,\alpha}(h_u;z) - \tau_{m,\alpha}(h;z)| &\leq (1-\alpha)\sum_{t=0}^{m-1}\binom{m-1}{t}\left|z^t(1-z)^{m-t-1}\right| \\ &\quad \times\left[\left(1-\frac{t}{m-1}\right)\|h_u-h\|_{r_1} + \frac{t}{m-1}\|h_n-h\|_{r_1}\right] \\ &\quad +\alpha\sum_{t=0}^{m}\binom{m}{t}\left|z^t(1-z)^{m-t}\right|\|h_u-h\|_{r_1} \\ |\tau_{m,\alpha}(h_u;z) - \tau_{m,\alpha}(h;z)| &\leq M_{m,\alpha,r_1}\|h_u-h\|_{r_1} \end{aligned}$$

Now, taking limit as $u \to \infty$, we get

$$\lim_{u\to\infty}|\tau_{m,\alpha}(h_u;z) - \tau_{m,\alpha}(h;z)| \leq \lim_{u\to\infty} M_{m,\alpha,r_1}||h_u - h||_{r_1}$$

Since

$$h(z) = \sum_{a=0}^{\infty} d_a z^a$$

is uniformly convergent on each compact subset of D_{R_1}, We have

$$\lim_{u\to\infty} ||h_u - h||_{r_1} = 0 \text{ for } |z| \leq r_1 < R_1.$$

So we have

$$\begin{aligned} \lim_{u\to\infty}|\tau_{m,\alpha}(h_u;z) - \tau_{m,\alpha}(h;z)| &\leq 0 \\ \lim_{u\to\infty}\tau_{m,\alpha}(h_u;z) &= \tau_{m,\alpha}(h;z) \end{aligned}$$

which completes the proof.

Example. Plot the graph of the α-Bernstein operator at $n = 5, n = 15$ and function is given by $f(z) = z^3$ where $\alpha = 0.3$.

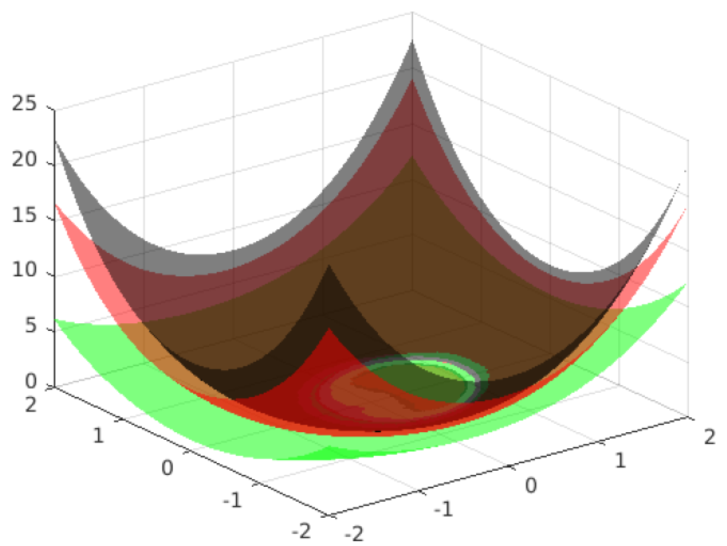

Figure 1: α-Bernstein operator at $n = 5$, $n = 15$ and function f

Result

Theorem 2.4. *Suppose that α is a real parameter satisfying the condition $0 \le \alpha \le 1$ and $h : D_{R_1} \to R$ is analytic in D_{R_1} , $R_1 > 1$ with $h(z) = \sum_{a=0}^{\infty} d_a z^a$*

1. *If $1 \le r_1 \le R_1$ is arbitrary fixed, then for all $|z| \le r_1$ and $m \in N$, we have*

$$|\tau_{m,\alpha}(h;z) - h(z)| \le \frac{M_{r_1}(h)}{m},$$

Where

$$0 < M_{r_1}(h) = \frac{3(1+r_1)}{2}\sum_{a=1}^{\infty}|d_a|(a+1)ar_1^{a-1} + (1+r_1)\sum_{a=1}^{\infty}|d_a|ar_1^{a-1} + 2r_1^2\sum_{a=2}^{\infty}|d_a|a(a-1)r_1^{a-2} < \infty.$$

2. *Also, if $1 \le r_1 < r_2 < R$, then for all $|z| \le r_1$ and $m, q \in N$, we have*

$$\left|\tau_{m,\alpha}^{(q)}(h;z) - h^{(q)}(z)\right| \le \frac{M_{r_2}(h)q!r_2}{m(r_2-r_1)^{q+1}},$$

where $M_{r_2}(h)$ is given by

$$0 < M_{r_2}(h) = \frac{3(1+r_2)}{2}\sum_{a=1}^{\infty}|d_a|(a+1)ar_2^{a-1} + (1+r_2)\sum_{a=1}^{\infty}|d_a|ar_2^{a-1} + 2r_2^2\sum_{a=2}^{\infty}|d_a|a(a-1)r_2^{a-2} < \infty.$$

Proof. (1) By Lemma 2.1, we can write

$$|\tau_{m,\alpha}(h;z) - h(z)| \le \sum_{a=0}^{\infty}|d_a||\tau_{m,\alpha}(l_a;z) - l_a(z)|$$

In order to estimate $|\tau_{m,\alpha}; l_a; z - la(z)|$ for fixed $m \in N$, we consider the following cases: $0 \le a \le m$ and $a > m$.

For $0 \le a \le m$ we can write

$$\tau_{m,\alpha}(l_a;z) - l_a(z) = (1-\alpha)\sum_{t=0}^{m-1}\binom{m-1}{t}z^t(1-z)^{m-t-1}\left[\left(1-\frac{t}{m-1}\right)\left(\frac{t}{m}\right)^a + \frac{t}{m-1}\left(\frac{t+1}{m}\right)^a\right] + \alpha\sum_{t=0}^{m}\binom{m}{t}z^t(1-z)^{m-t}\left(\frac{t}{m}\right)^a - \left(\frac{t}{m}\right)^a$$

Now when $a = 0$, we have

$$\tau_{m,\alpha}(l_0;z) - l_0(z) = (1-\alpha)\sum_{t=0}^{m-1}\binom{m-1}{t}z^t(1-z)^{m-t-1}\left[\left(1-\frac{t}{m-1}\right)\left(\frac{t}{m}\right)^0 + \frac{t}{m-1}\left(\frac{t+1}{m}\right)^0\right] + \alpha\sum_{t=0}^{m}\binom{m}{t}z^t(1-z)^{m-t}\left(\frac{t}{m}\right)^0 - \left(\frac{t}{m}\right)^0$$

$$\begin{aligned} \tau_{m,\alpha}(l_0;z)-l_0(z) &= (1-\alpha)\sum_{t=0}^{m-1}\binom{m-1}{t}z^t(1-z)^{m-t-1}\left[1-\frac{t}{m-1}+\frac{t}{m-1}\right] \\ &+\alpha\sum_{t=0}^{m}\binom{m}{t}z^t(1-z)^{m-t}-1 \\ \tau_{m,\alpha}(l_0;z)-l_0(z) &= (1-\alpha)\sum_{t=0}^{m-1}\binom{m-1}{t}z^t(1-z)^{m-t-1}+\alpha\sum_{t=0}^{m}\binom{m}{t}z^t(1-z)^{m-t-1} \end{aligned} \tag{13}$$

Now since

$$B_m(1;z)=\sum_{a=0}^{m}\binom{m}{a}z^a(1-z)^{m-a}=1$$

Substituting above value in equation (13), we get

$$\begin{aligned} \tau_{m,\alpha}(l_0;z)-l_0(z) &= (1-\alpha)(1)+\alpha(1)-1 \\ \tau_{m,\alpha}(l_0;z)-l_0(z) &= 1-\alpha+\alpha-1 \\ \tau_{m,\alpha}(l_0;z)-l_0(z) &= 0 \end{aligned}$$

Now if $a = 1$, then we have

$$\begin{aligned} \tau_{m,\alpha}(e_1;z)-e_1(z) &= (1-\alpha)\sum_{t=0}^{m-1}\binom{m-1}{t}z^t(1-z)^{m-t-1}\left[\left(1-\frac{t}{m-1}\right)\left(\frac{t}{m}\right)^1\right. \\ &\left.+\frac{t}{m-1}\left(\frac{t+1}{m}\right)^1\right]+\alpha\sum_{t=0}^{m}\binom{m}{t}z^t(1-z)^{m-t}\left(\frac{t}{m}\right)^1-\left(\frac{t}{m}\right)^1 \end{aligned}$$

$$\begin{aligned} \tau_{m,\alpha}(l_1;z)-l_1(z) &= (1-\alpha)\sum_{t=0}^{m-1}\binom{m-1}{t}z^t(1-z)^{m-t-1}\left[\left(1-\frac{t}{m-1}\right)\left(\frac{t}{m}\right)\right. \\ &\left.+\frac{t}{m-1}\left(\frac{t+1}{m}\right)\right]+\alpha\sum_{t=0}^{m}\binom{m}{t}z^t(1-z)^{m-t}\left(\frac{t}{m}\right)-\left(\frac{t}{m}\right) \end{aligned}$$

$$\begin{aligned} \tau_{m,\alpha}(l_1;z)-l_1(z) &= (1-\alpha)\sum_{t=0}^{m-1}\binom{m-1}{t}z^t(1-z)^{m-t-1}\left[\frac{t}{m}-\frac{t^2}{m(m-1)}\right. \\ &\left.+\frac{t^2}{m(m-1)}+\frac{t}{m(m-1)}\right]+\alpha\sum_{t=0}^{m}\binom{m}{t}z^t(1-z)^{m-t}\left(\frac{t}{m}\right)-\left(\frac{t}{m}\right) \end{aligned}$$

$$\begin{aligned} \tau_{m,\alpha}(l_1;z)-l_1(z) &= (1-\alpha)\sum_{t=0}^{m-1}\binom{m-1}{t}z^t(1-z)^{m-t-1}\left[\frac{t}{m}+\frac{t}{m(m-1)}\right] \\ &+\alpha\sum_{t=0}^{m}\binom{m}{t}z^t(1-z)^{m-t}\left(\frac{t}{m}\right)-\left(\frac{t}{m}\right) \end{aligned}$$

$$\begin{aligned}\tau_{m,\alpha}(l_1;z)-l_1(z) &= (1-\alpha)\sum_{t=0}^{m-1}\binom{m-1}{t}z^t(1-z)^{m-t-1}\left[\frac{t(m-1)+t}{m(m-1)}\right]\\ &+\alpha\sum_{t=0}^{m}\binom{m}{t}z^t(1-z)^{m-t}\left(\frac{t}{m}\right)-\left(\frac{t}{m}\right)\end{aligned}$$

$$\begin{aligned}\tau_{m,\alpha}(l_1;z)-l_1(z) &= (1-\alpha)\sum_{t=0}^{m-1}\binom{m-1}{t}z^t(1-z)^{m-t-1}\left[\frac{mt-t+t}{m(m-1)}\right]\\ &+\alpha\sum_{t=0}^{m}\binom{m}{t}z^t(1-z)^{m-t}\left(\frac{t}{m}\right)-\left(\frac{t}{m}\right)\end{aligned}$$

$$\begin{aligned}\tau_{m,\alpha}(l_1;z)-l_1(z) &= (1-\alpha)\sum_{t=0}^{m-1}\binom{m-1}{t}z^t(1-z)^{m-t-1}\left(\frac{mt}{m(m-1)}\right)\\ &+\alpha\sum_{t=0}^{m}\binom{m}{t}z^t(1-z)^{m-t}\left(\frac{t}{m}\right)-\left(\frac{t}{m}\right)\end{aligned}$$

$$\begin{aligned}\tau_{m,\alpha}(l_1;z)-l_1(z) &= (1-\alpha)\sum_{t=0}^{m-1}\binom{m-1}{t}z^t(1-z)^{m-t-1}\left(\frac{t}{m-1}\right)\\ &+\alpha\sum_{t=0}^{m}\binom{m}{t}z^t(1-z)^{m-t}\left(\frac{t}{m}\right)-\left(\frac{t}{m}\right)\\ \tau_{m,\alpha}(l_1;z)-l_1(z) &= \Gamma+\alpha\sum_{t=0}^{m}\binom{m}{t}z^t(1-z)^{m-t}\left(\frac{t}{m}\right)-\left(\frac{t}{m}\right) \qquad (14)\end{aligned}$$

Where

$$\Gamma=(1-\alpha)\sum_{t=0}^{m-1}\binom{m-1}{t}z^t(1-z)^{m-t-1}\left(\frac{t}{m-1}\right)$$

Since

$$\binom{m-1}{m}=0$$

So, the value of Γ at $t = m$ is zero. Replacing $m - 1$ by m in Γ, we get

$$\Gamma=(1-\alpha)\sum_{t=0}^{m}\binom{m}{t}z^t(1-z)^{m-t}\left(\frac{t}{m}\right)$$

Substituting value of Γ in equation (14), we get

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$$\begin{aligned} \tau_{m,\alpha}(l_1;z) - l_1(z) &= (1-\alpha)\sum_{t=0}^{m}\binom{m}{t}z^t(1-z)^{m-t}\left(\frac{t}{m}\right) \\ &+\alpha\sum_{t=0}^{m}\binom{m}{t}z^t(1-z)^{m-t}\left(\frac{t}{m}\right) - \left(\frac{t}{m}\right) \quad (15) \end{aligned}$$

Now since

$$B_m(1;z) = \sum_{t=0}^{m}\binom{m}{t}z^t(1-z)^{m-t} = 1$$

Substituting above value in equation (15)

$$\begin{aligned} \tau_{m,\alpha}(l_1;z) - l_1(z) &= (1-\alpha)(1)\left(\frac{t}{m}\right) + \alpha(1)\left(\frac{t}{m}\right) - \left(\frac{t}{m}\right) \\ \tau_{m,\alpha}(l_1;z) - l_1(z) &= (1-\alpha)\left(\frac{t}{m}\right) + \alpha\left(\frac{t}{m}\right) - \left(\frac{t}{m}\right) \\ \tau_{m,\alpha}(l_1;z) - l_1(z) &= \left(\frac{t}{m}\right) - \alpha\left(\frac{t}{m}\right) + \alpha\left(\frac{t}{m}\right) - \left(\frac{t}{m}\right) \\ \tau_{m,\alpha}(l_1;z) - l_1(z) &= 0 \end{aligned}$$

For $a = 0$ and 1, we have $\tau_{m,\alpha}(la;\ z) - la(z) = 0$. So we consider the case $2 \le a \le m$. Denoting

$$\xi_{m,\alpha,a} = \tau_{m,\alpha}(l_a;z)$$

by the recurrence formula in (12) we get

$$\begin{aligned} \xi_{m,\alpha,a}(z) &= \frac{z(1-z)}{m}[\xi_{m,\alpha,a-1}(z)]' + \left[z + \frac{1-z}{m}\right][\xi_{m,\alpha,a-1}(z)] + \frac{(1-\alpha)}{m}\left(\frac{m-1}{m}\right)^{a-1} \\ &\times[B_{m-1}(l_a;z) - B_{m-1}(l_{a-1};z)] - \frac{\alpha(1-z)}{m}B_m(l_{a-1};z) \end{aligned}$$

Adding and Subtracting

$$\frac{(a-1)z^{a-1}(1-z)}{m},\ z^a \text{ and } \frac{(1-z)z^{a-1}}{m}$$

$$\begin{aligned} \xi_{m,\alpha,a}(z) &= \frac{z(1-z)}{m}[\xi_{m,\alpha,a-1}(z)]' - \frac{(a-1)z^{a-1}(1-z)}{m} + \frac{(a-1)z^{a-1}(1-z)}{m} \\ &+ \left[z + \frac{1-z}{m}\right][\xi_{m,\alpha,a-1}(z)] - z^a + z^a - \frac{(1-z)}{m}z^{a-1} + \frac{(1-z)}{m}z^{a-1} \\ &+ \frac{(1-\alpha)}{m}\left(\frac{m-1}{m}\right)^{a-1}[B_{m-1}(l_a;z) - B_{m-1}(l_{a-1};z)] \\ &- \frac{\alpha(1-z)}{m}B_m(l_{a-1};z) + \frac{a(1-z)}{m}z^{a-1} \\ \xi_{m,\alpha,a}(z) - z^a &= \frac{z(1-z)}{m}[\xi_{m,\alpha,a-1}(z) - z^{a-1}]' + \left[z + \frac{1-z}{m}\right][\xi_{m,\alpha,a-1}(z) - z^{a-1}] \\ &+ \frac{(1-\alpha)}{m}\left(\frac{m-1}{m}\right)^{a-1}[B_{m-1}(l_a;z) - B_{m-1}(l_{a-1};z)] \\ &- \frac{\alpha(1-z)}{m}B_m(l_{a-1};z) + \frac{(a-1)(1-z)}{n}z^{a-1} + \frac{(1-z)}{m}z^{a-1} \end{aligned}$$

$$\begin{aligned}\xi_{m,\alpha,a}(z) - z^a &= \frac{z(1-z)}{m}[\xi_{m,\alpha,a-1}(z) - z^{a-1}]' + \left[z + \frac{1-z}{m}\right][\xi_{m,\alpha,a-1}(z) - z^{a-1}] \\ &+ \frac{(1-\alpha)}{n}\left(\frac{m-1}{m}\right)^{a-1}[B_{m-1}(l_a;z) - B_{m-1}(l_{a-1};z)] \\ &- \frac{\alpha(1-z)}{m}B_m(l_{a-1};z) + \frac{(1-z)}{m}z^{a-1}(a-1+1) \\ \xi_{m,\alpha,a}(z) - z^a &= \frac{z(1-z)}{m}[\xi_{m,\alpha,a-1}(z) - z^{a-1}]' \\ &+ \left[z + \frac{1-z}{m}\right][\xi_{m,\alpha,a-1}(z) - z^{a-1}] \\ &+ \frac{(1-\alpha)}{m}\left(\frac{m-1}{m}\right)^{a-1}[B_{m-1}(l_a;z) - B_{m-1}(l_{a-1};z)] \\ &- \frac{\alpha(1-z)}{m}B_m(l_{a-1};z) + \frac{a(1-z)}{m}z^{a-1}.\end{aligned}$$

Using the Bernstein's inequality

$$|P_m'(z)| \leq \frac{m}{r_1}\|P_m\|_{r_1}$$

[14] in the above recurrence, for $|z| \leq r_1, r_1 \geq 1$, we have

$$\begin{aligned}|\xi_{m,\alpha,a}(z) - l_a(z)| &\leq (a-1)\frac{1+r_1}{m}\|\xi_{m,\alpha,a-1}(z) - l_{a-1}(z)\|_{r_1} \\ &+ \left[r_1 + \frac{(1+r_1)}{m}\right]|\xi_{m,\alpha,a-1}(z) - l_{a-1}(z)| \\ &+ \frac{(1-\alpha)}{m}\left(\frac{m-1}{m}\right)^{a-1}[|B_{m-1}(l_a;z)| + |B_{m-1}(l_{a-1;z})|] \\ &+ \frac{\alpha(1+r_1)}{m}|B_m(l_{a-1;z})| + \frac{a(1+r_1)}{m}r_1^{a-1}\end{aligned}$$

According to the inequality [22]

$$|\xi_a(z)| \leq R_1^a \qquad |z-c| \leq R_1$$

By some calculations the last inequality follows that

$$\begin{aligned}|\xi_{m,\alpha,a}(z) - l_a(z)| &\leq (a-1)\frac{1+r_1}{m}\|\xi_{m,\alpha,a-1}(z) - l_{a-1}(z)\|_{r_1} \\ &+ \left[r_1 + \frac{(1+r_1)}{m}\right]|\xi_{m,\alpha,a-1}(z) - l_{a-1}(z)| \\ &+ \frac{(1-\alpha)}{m}[r_1^a + r_1^{a-1}] + \frac{\alpha(1+r_1)}{m}r_1^{a-1} + \frac{a(1+r_1)}{m}r_1^{a-1}\end{aligned}$$

$$\begin{aligned}
|\xi_{m,\alpha,a}(z)-l_a(z)| &\leq (a-1)\frac{1+r_1}{m}\|\xi_{m,\alpha,a-1}(z)-l_{a-1}(z)\|_{r_1}\\
&+\left[r_1+\frac{(1+r_1)}{m}\right]|\xi_{m,\alpha,a-1}(z)-l_{a-1}(z)|\\
&+\frac{(1-\alpha)}{m}r_1^a+\frac{(1-\alpha)}{m}r_1^{a-1}+\frac{\alpha(1+r_1)}{m}r_1^{a-1}+\frac{a(1+r_1)}{m}r_1^{a-1}\\
|\xi_{m,\alpha,a}(z)-l_a(z)| &\leq (a-1)\frac{1+r_1}{m}\|\xi_{m,\alpha,a-1}(z)-l_{a-1}(z)\|_{r_1}\\
&+\left[r_1+\frac{(1+r_1)}{m}\right]|\xi_{m,\alpha,a-1}(z)-l_{a-1}(z)|\\
&+[(1-\alpha)r_1+(1-\alpha)+\alpha(1+r_1)+a(1+r_1)]\frac{r_1^{a-1}}{m}\\
|\xi_{m,\alpha,a}(z)-l_a(z)| &\leq (a-1)\frac{1+r_1}{m}\|\xi_{m,\alpha,a-1}(z)-l_{a-1}(z)\|_{r_1}\\
&+\left[r_1+\frac{(1+r_1)}{m}\right]|\xi_{m,\alpha,a-1}(z)-l_{a-1}(z)|\\
&+[r_1-\alpha r_1+1-\alpha+\alpha+\alpha r_1+a(1+r_1)]\frac{r_1^{a-1}}{m}\\
|\xi_{m,\alpha,a}(z)-l_a(z)| &\leq (a-1)\frac{1+r_1}{m}\|\xi_{m,\alpha,a-1}(z)-l_{a-1}(z)\|_{r_1}\\
&+\left[r_1+\frac{(1+r_1)}{m}\right]|\xi_{m,\alpha,a-1}(z)-l_{a-1}(z)|\\
&+[r_1+1+a(1+r_1)]\frac{r_1^{a-1}}{m}\\
|\xi_{m,\alpha,a}(z)-l_a(z)| &\leq (a-1)\frac{1+r_1}{m}\|\xi_{m,\alpha,a-1}(z)-l_{a-1}(z)\|_{r_1}\\
&+\left[r_1+\frac{(1+r_1)}{m}\right]|\xi_{m,\alpha,a-1}(z)-l_{a-1}(z)|+[(r_1+1)(a+1)]\frac{r_1^{a-1}}{m}\\
|\xi_{m,\alpha,a}(z)-l_a(z)| &\leq (a-1)\frac{1+r_1}{m}\|\xi_{m,\alpha,a-1}(z)-l_{a-1}(z)\|_{r_1}+r_1|\xi_{m,\alpha,a-1}(z)-l_{a-1}(z)|\\
&+\frac{(1+r_1)}{m}\|\xi_{m,\alpha,a-1}(z)-l_{a-1}(z)\|_{r_1}+\frac{(a+1)(1+r_1)}{m}r_1^{a-1}\\
|\xi_{m,\alpha,a}(z)-l_a(z)| &\leq r_1|\xi_{m,\alpha,a-1}(z)-l_{a-1}(z)|+\frac{a(1+r_1)}{m}\{\|\xi_{m,\alpha,a-1}\|_{r_1}+\|l_{a-1}\|_{r_1}\}\\
&+\frac{(a+1)(1+r_1)}{m}r_1^{a-1}\\
|\xi_{m,\alpha,a}(z)-l_a(z)| &\leq r_1|\xi_{m,\alpha,a-1}(z)-l_{a-1}(z)|+\frac{a(1+r_1)}{m}\|\xi_{m,\alpha,a-1}\|_{r_1}+\frac{a(1+r_1)}{m}\|l_{a-1}\|_{r_1}\\
&+\frac{(a+1)(1+r_1)}{m}r_1^{a-1}
\end{aligned}$$

According to the inequality

$$\|l_{a-1}(z)\|_{r_1}\leq r_1^{a-1}\qquad |z-c|\leq r_1$$

By some calculations the last inequality follows that

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$$\begin{aligned}|\xi_{m,\alpha,a}(z)-l_a(z)| &\leq r_1|\xi_{m,\alpha,a-1}(z)-l_{a-1}(z)|+\frac{a(1+r_1)}{m}\|\xi_{m,\alpha,a-1}\|_{r_1}+\frac{a(1+r_1)}{m}r_1^{a-1}\\&+\frac{(a+1)(1+r_1)}{m}r_1^{a-1} \qquad (16)\\|\xi_{m,\alpha,a}(z)-l_a(z)| &\leq r_1|\xi_{m,\alpha,a-1}(z)-l_{a-1}(z)|+\frac{a(1+r_1)}{m}\|\xi_{m,\alpha,a-1}\|_{r_1}\\&+(a+1+a)\frac{(1+r_1)}{m}r_1^{a-1}\\|\xi_{m,\alpha,a}(z)-l_a(z)| &\leq r_1|\xi_{m,\alpha,a-1}(z)-l_{a-1}(z)|+\frac{a(1+r_1)}{m}\|\xi_{m,\alpha,a-1}\|_{r_1}\\&+\frac{(2a+1)(1+r_1)}{m}r_1^{a-1} \qquad (17)\end{aligned}$$

Now, let us obtain an upper estimate for $\left||\xi_{m,\alpha,a}|\right|_{r_1}$. For this aim, using the operator's representation via divided difference in (11)

$$\begin{aligned}|\xi_{m,\alpha,a}(z)| &= \sum_{q=0}^{m}\left[(1-\alpha)\binom{m-1}{q}\frac{q!}{n^q}\left\{\left[0,\frac{1}{m},\cdots,\frac{q}{m};l_a\right]\right.\right.\\&\left.\left.+\frac{q}{m-1}\left[\frac{1}{m},\frac{2}{m},\cdots,\frac{1+q}{m};l_a\right]\right\}+\alpha\binom{m}{q}\frac{q!}{m^q}\left[0,\frac{1}{m},\cdots,\frac{q}{m};l_a\right]\right]z^q\end{aligned}$$

Now since $|z|\leq r_1$ and $a > m$, taking summation from $q = 0$ to a, we get

$$\begin{aligned}|\xi_{m,\alpha,a}(z)| &\leq \sum_{q=0}^{a}\left[(1-\alpha)\binom{m-1}{q}\frac{q!}{m^q}\left\{\left[0,\frac{1}{m},\cdots,\frac{q}{m};l_a\right]\right.\right.\\&\left.\left.+\frac{q}{m-1}\left[\frac{1}{m},\frac{2}{m},\cdots,\frac{1+q}{m};l_a\right]\right\}+\alpha\binom{m}{q}\frac{q!}{m^q}\left[0,\frac{1}{m},\cdots,\frac{q}{m};l_a\right]\right]r_1^q\\|\xi_{m,\alpha,a}(z)| &\leq r_1^a\left(\sum_{q=0}^{a}\left[(1-\alpha)\binom{m-1}{q}\frac{q!}{m^q}\left\{\left[0,\frac{1}{m},\cdots,\frac{q}{m};l_a\right]\right.\right.\right.\\&\left.\left.\left.+\frac{q}{m-1}\left[\frac{1}{m},\frac{2}{m},\cdots,\frac{1+q}{m};l_a\right]\right\}+\alpha\binom{m}{q}\frac{q!}{m^q}\left[0,\frac{1}{m},\cdots,\frac{q}{m};l_a\right]\right]\right)\\|\xi_{m,\alpha,a}(z)| &= r_1^a\tau_{m,\alpha}(l_a;1) \qquad (18)\end{aligned}$$

For $|z|\leq r_1$ with $1\leq r1 < R1$. Since $\tau_{m,\alpha}(h;\ 1) = h(1)$ for the α-Bernstein operator, (18) reduces to

$$|\xi_{m,\alpha,a}(z)|\leq r_1^a$$

Substituting the last inequality into (17), the following can be easily obtained

$$\begin{aligned}|\xi_{m,\alpha,a}(z)-l_a(z)| &\leq r_1|\xi_{m,\alpha,a-1}(z)-l_{a-1}(z)|+\frac{a(1+r_1)}{m}r_1^{a-1}+\frac{(2a+1)(1+r_1)}{m}r_1^{a-1}\\|\xi_{m,\alpha,a}(z)-l_a(z)| &\leq r_1|\xi_{m,\alpha,a-1}(z)-l_{a-1}(z)|+(2a+1+a)\frac{(1+r_1)}{m}r_1^{a-1}\\|\xi_{m,\alpha,a}(z)-l_a(z)| &\leq |\xi_{m,\alpha,a-1}(z)-l_{a-1}(z)|+\frac{(3a+1)(1+r_1)}{m}r_1^{a-1} \qquad (19)\end{aligned}$$

By writing $a = 1, 2, ...,$ in (19), step by step, one has

$$|\xi_{m,\alpha,a}(z) - l_a(z)| \leq \sum_{s=1}^{a}(3s+1)\frac{(1+r_1)}{m}r_1^{a-1}$$

$$|\xi_{m,\alpha,a}(z) - l_a(z)| \leq \left[\frac{3a(a+1)}{2}+a\right]\frac{(1+r_1)}{m}r_1^{a-1}$$

Case 2 : For $a > m \geq 2$ and $|z| \leq r1 < R1$, from (11) we have

$$|\xi_{m,\alpha,a}(z) - l_a(z)| \leq |\xi_{m,\alpha,a}(z)| + |l_a(z)|$$

$$|\xi_{m,\alpha,a}(z) - l_a(z)| \leq \sum_{q=0}^{m}\left[(1-\alpha)\binom{m-1}{q}\frac{q!}{m^q}\left\{\left[0,\frac{1}{m},\cdots,\frac{q}{m};l_a\right] + \frac{q}{m-1}\left[\frac{1}{m},\frac{2}{m},\cdots,\frac{1+q}{m};l_a\right]\right\} + \alpha\binom{m}{q}\frac{q!}{m^q}\left[0,\frac{1}{m},\cdots,\frac{q}{m};l_a\right]\right]z^q + z^a$$

Since$|z| \leq r1$, So we obtain

$$|\xi_{m,\alpha,a}(z) - l_a(z)| \leq \sum_{q=0}^{m}\left[(1-\alpha)\binom{m-1}{q}\frac{q!}{m^q}\left\{\left[0,\frac{1}{m},\cdots,\frac{q}{m};l_a\right] + \frac{q}{m-1}\left[\frac{1}{m},\frac{2}{m},\cdots,\frac{1+q}{m};l_a\right]\right\} + \alpha\binom{m}{q}\frac{q!}{m^q}\left[0,\frac{1}{m},\cdots,\frac{q}{m};l_a\right]\right]r_1^q + r_1^a$$

$$|\xi_{m,\alpha,a}(z) - l_a(z)| \leq r_1^m\left(\sum_{q=0}^{a}\left[(1-\alpha)\binom{m-1}{q}\frac{q!}{m^q}\left\{\left[0,\frac{1}{m},\cdots,\frac{q}{m};l_a\right] + \frac{q}{m-1}\left[\frac{1}{m},\frac{2}{m},\cdots,\frac{1+q}{m};l_a\right]\right\} + \alpha\binom{m}{q}\frac{q!}{m^q}\left[0,\frac{1}{m},\cdots,\frac{q}{m};l_a\right]\right]\right) + r_1^a$$

$$|\xi_{m,\alpha,a}(z) - l_a(z)| = r_1^m\tau_{m,\alpha}(l_a;1) + r_1^a$$

Using the fact $\tau_{m,\alpha}(h;\ 1) = h(1)$ for the α-Bernstein operator, the last inequality gives

$$|\xi_{m,\alpha,a}(z) - l_a(z)| \leq r_1^m + r_1^a \leq 2r_1^a \qquad (since\ \ m < a)$$

$$|\xi_{m,\alpha,a}(z) - l_a(z)| \leq 2(m-1)r_1^a$$

$$|\xi_{m,\alpha,a}(z) - l_a(z)| \leq \frac{2a(a-1)}{m}r_1^a$$

As a consequence, combining Case 2 with the above Case 1, we obtain

$$|\xi_{m,\alpha,a}(z) - l_a(z)| \leq \left[\frac{3a(a+1)}{2}+a\right]\frac{(1+r_1)}{m}r_1^{a-1} + \frac{2a(a-1)}{m}r_1^a \qquad (20)$$

By writing $a = 1,2,\_\_\_$ in (20), step by step, one has

$$|\xi_{m,\alpha,a}(z) - l_a(z)| \leq \frac{3(1+r_1)}{2m}\sum_{a=1}^{\infty}|d_a|(a+1)ar_1^{a-1}\frac{(1+r_1)}{m}\sum_{a=1}^{\infty}|d_a|ar_1^{a-1} + \frac{2r_1^2}{m}\sum_{a=1}^{\infty}|d_a|a(a-1)r_1^{a-2}$$

(2) For the simultaneous approximation, denoting by ζ the circle of radius $r_2 > r_1$ and center 0, since for any $|z| \leq r_1$ and $\in \zeta$, we have $|u-z| \geq r_2 - r_1$, by Cauchy's formulas it follows that for all $|z| \leq r_1$ and $m \in N$, we have

$$\left|\tau_{m,\alpha}^{(q)}(h;z) - h^{(q)}(z)\right| = \left|\frac{q!}{2\pi\iota}\int_{\zeta}\frac{\tau_{m,\alpha}(h;u)-h(u)}{u-z|^{q+1}}du\right|$$

$$\left|\tau_{m,\alpha}^{(q)}(h;z) - h^{(q)}(z)\right| \leq \frac{q!}{2\pi}\int_{\zeta}\frac{|\tau_{m,\alpha}(h;u)-h(u)|}{|u-z|^{q+1}}|du|$$

$$\left|\tau_{m,\alpha}^{(q)}(h;z) - h^{(q)}(z)\right| \leq \frac{q!}{2\pi}\frac{M_{r_2}(h)}{m}\frac{2\pi r_2}{(r_2-r_1)^{q+1}}$$

$$\left|\tau_{m,\alpha}^{(q)}(h;z) - h^{(q)}(z)\right| \leq \frac{M_{r_2}(h)q!r_2}{m(r_2-r_1)^{q+1}}$$

which proves (2) and the theorem.

Voronovskaja Type

Theorem 2.5. *Let r_1 and R_1 be constants such that $1 \leq r_1 < R_1$. Also, $h: D_{R_1} \to \mathbb{C}$ is analytic in 1 , with $h(z) = \sum_{a=0}^{\infty} d_a z^a$ Then for all $m \in N$ and $\alpha \in [0;\ 1]$, we have*

$$\lim_{m\to\infty} m[\tau_{m,\alpha}(h;z) - h(z)] = \frac{z(1-z)}{2}h"(z)$$

uniformly in $\overline{D_{r_1}}$.

Proof. Firstly we will prove the result [10]

$$\lim_{m\to\infty}\left(m[\tau_{m,\alpha}(h;y) - h(y)] - \frac{y(1-y)}{2}h''(y)\right) = 0$$

for each $y \in [0;\ 1]$

For $t \leq m$ we have the taylor's formula that

$$h(s) = h(y) + (s-y)h'(y) + \frac{1}{2}(s-y)h''(y) + o(s)(s-y)^2$$

where $lims \to y\ o(s) = 0$. Setting $s = \frac{t}{m}$, we get

$$h\left(\frac{t}{m}\right) = h(y) + \left(\frac{t}{m}-y\right)h'(y) + \frac{1}{2}\left(\frac{t}{m}-y\right)^2 h''(y) + o\left(\frac{t}{m}\right)\left(\frac{t}{m}-y\right)^2 \tag{21}$$

Thus we have

$$m[\tau_{m,\alpha}(h;y) - h(y)] = m\sum_{t=0}^{m} q_{m,t}^{(\alpha)}(y)\left[h\left(\frac{t}{m}\right) - h(y)\right] \qquad (22)$$

By equations (21) and (22), we have

$$m[\tau_{m,\alpha}(h;y) - h(y)] = m\sum_{t=0}^{m} q_{m,t}^{(\alpha)}(y)\left[\left(\frac{t}{m} - y\right)h'(y) + \frac{1}{2}\left(\frac{t}{m} - y\right)^2 h''(y) + o\left(\frac{t}{m}\right)\left(\frac{t}{m} - y\right)^2\right] \qquad (23)$$

Now if we suppose that

$$O_l(y) = \sum_{t=0}^{m}(t - my)^l q_{m,t}^{(\alpha)}(y), \quad l = 1, 2$$

So,

$$O_1(y) = \sum_{t=0}^{m}(t - my)q_{m,t}^{\alpha}(y) \qquad (24)$$

And

$$O_2(y) = \sum_{t=0}^{m}(t - my)^2 q_{m,t}^{\alpha}(y) \qquad (25)$$

Substituting the values of O1 and O2 in equation (23), we have

$$m[\tau_{m,\alpha}(h;y) - h(y)] = O_1(y)h'(y) + \frac{1}{2m}O_2(y)h''(y) + m\sum_{t=0}^{m} o\left(\frac{t}{m}\right)\left(\frac{t}{m} - y\right)^2 q_{m,t}^{\alpha}(y) \qquad (26)$$

Now we will calculate the value of O1 and O2. We have the following relations that

$$\sum_{t=0}^{m} q_{m,t}^{\alpha}(y) = 1 \qquad (27)$$

$$\sum_{t=0}^{m} t q_{m,t}^{\alpha}(y) = my \qquad (28)$$

$$\sum_{t=0}^{m} t^2 q_{m,t}^{\alpha}(y) = m^2y^2 + [m + 2(1-\alpha)]y(1-y) \qquad (29)$$

From equation (24), we have

$$\begin{aligned} O_1(y) &= \sum_{t=0}^{m}(t - my)q_{m,t}^{\alpha}(y) \\ O_1(y) &= \sum_{t=0}^{m} t q_{m,t}^{\alpha}(y) - \sum_{t=0}^{m} my q_{m,t}^{\alpha}(y) \\ O_1(y) &= \sum_{t=0}^{m} t q_{m,t}^{\alpha}(y) - my\sum_{t=0}^{m} q_{m,t}^{\alpha}(y) \qquad (30) \end{aligned}$$

Substituting the values from equation (27) and (28) in equation (30)

$$\begin{aligned} O_1(y) &= my - my(1) \\ O_1(y) &= my - my \\ O_1(y) &= 0 \end{aligned} \tag{31}$$

Now from equation (25), we have

$$\begin{aligned} O_2(y) &= \sum_{t=0}^{m} (t-my)^2 q_{m,t}^{\alpha}(y) \\ O_2(y) &= \sum_{t=0}^{m} (t^2 - 2tmy + m^2y^2) q_{m,t}^{\alpha}(y) \\ O_2(y) &= \sum_{t=0}^{m} t^2 q_{m,t}^{\alpha}(y) - \sum_{t=0}^{m} 2tmy q_{m,t}^{\alpha}(y) + \sum_{t=0}^{m} m^2y^2 q_{m,t}^{\alpha}(y) \\ O_2(y) &= \sum_{t=0}^{m} t^2 q_{m,t}^{\alpha}(y) - 2my \sum_{t=0}^{m} t q_{m,t}^{\alpha}(y) + m^2y^2 \sum_{t=0}^{m} q_{m,t}^{\alpha}(y) \end{aligned} \tag{32}$$

Substituting the values from equation (27), (28) and (29) in equation (32)

$$\begin{aligned} O_2(y) &= m^2y^2 + [m + 2(1-\alpha)]y(1-y) - 2my(my) + m^2y^2 \\ O_2(y) &= m^2y^2 + [m + 2(1-\alpha)]y(1-y) - 2m^2y^2 + m^2y^2 \\ O_2(y) &= [m + 2(1-\alpha)]y(1-y) \end{aligned} \tag{33}$$

Then use of equation (31) and (33) in (26) gives

$$\begin{aligned} m[\tau_{m,\alpha}(h;y) - h(y)] &= \frac{1}{2m}[m + 2(1-\alpha)]y(1-y)h''(y) + m\sum_{t=0}^{m} o\left(\frac{t}{m}\right)\left(\frac{t}{m} - y\right)^2 q_{m,t}^{\alpha}(y) \\ m[\tau_{m,\alpha}(h;y) - h(y)] &= \left(\frac{1}{2} + \frac{1-\alpha}{m}\right) y(1-y)h''(y) + mR_m(y) \end{aligned}$$

Where

$$R_m(y) = \sum_{t=0}^{m} o\left(\frac{t}{m}\right)\left(\frac{t}{m} - y\right)^2 q_{m,t}^{\alpha}(y)$$

We can get the inequality for $0 \le \alpha \le 1$

$$|R_m(y)| = \sum_{|t/m-y|<m^{-1/8}} \left|o\left(\frac{t}{m}\right)\right| \left(\frac{t}{m} - y\right)^2 q_{m,t}^{\alpha}(y) + \sum_{|t/m-y|\ge m^{-1/8}} \left|o\left(\frac{t}{m}\right)\right| \left(\frac{t}{m} - y\right)^2 q_{m,t}^{\alpha}(y)$$

Let $\delta > 0$ be given. We can find m sufficiently large such that $\left|\frac{t}{m} - y\right| < m^{-1/8}$ implies $\left|o\left(\frac{t}{m}\right)\right| < \delta$. Hence

$$|R_m(y)| \le \frac{\delta}{m^2} O_2(y) + M \sum_{|t/m-y|\ge m^{-1/8}} q_{m,t}^{\alpha}(y)$$

where $M = \sup_{0\le s\le 1} o(s)(s-y)^2$. By use of equation (33) and the Lemma that there is a constant $C1$ independent of m such that for all $y \in [0,1]$ and any real $\lambda \in (0;\ 1/4)$ (where $\lambda = 1/8$),

$$\sum_{|t/m-y|\ge m^{-\lambda}} q_{m,t}^{\alpha}(y) \le C_1 m^{2(2\lambda-1)}$$

We have

$$\begin{aligned} |R_m(y)| &\leq \frac{\delta}{m^2}[m+2(1-\alpha)]y(1-y)+\frac{MC_1}{m^{1/2}} \\ |R_m(y)| &\leq m\left[1+\frac{2(1-\alpha)}{m}\right]y(1-y)+\frac{MC_1}{m^{1/2}} \end{aligned}$$

where δ is arbitrary and for the $\lim m \to \infty$ the values of right hand side of the above equation will be zero. So we have

$$\lim_{m\to\infty}\left(m[\tau_{m,\alpha}(h;y)-h(y)]-\frac{y(1-y)}{2}h''(y)\right)=0$$

for each $y \in [0; 1]$.

According to the classical Vitali's result [14], it suffices to show that the sequence

$$\left\{m[\tau_{m,\alpha}(h;z)-h(z)]-\frac{z(1-z)}{2}h''(z)\right\}_{m\in\mathbb{N}}$$

of analytic functions in $DR1$ is bounded in each $Dr1$. By Theorem 2.4, we can write

$$\left|m[\tau_{m,\alpha}(h;z)-h(z)]-\frac{z(1-z)}{2}h''(z)\right| \leq m|\tau_{m,\alpha}(h;z)-h(z)|+\left|\frac{z(1-z)}{2}h''(z)\right|$$

Now since

$$\|h''(z)\|_{r_1}=\max\{|h'(z)| \text{ where } |z|\leq r_1\}$$

$$\begin{aligned} \left|m[\tau_{m,\alpha}(h;z)-h(z)]-\frac{z(1-z)}{2}h''(z)\right| &\leq m|\tau_{m,\alpha}(h;z)-h(z)|+\frac{r_1(1+r_1)}{2}\|h''(z)\|_{r_1} \\ \left|m[\tau_{m,\alpha}(h;z)-h(z)]-\frac{z(1-z)}{2}h''(z)\right| &\leq M_{r_1}(h)+\frac{r_1(1+r_1)}{2}\|h''\|_{r_1}, \end{aligned}$$

for all $z \in Dr1$ with $1 \leq r1 < R1$ and $\alpha \in [0,1]$, where $Mr1(h)$ is the constant

$$\begin{aligned} 0<M_{r_1}(h) &= \frac{3(1+r_1)}{2}\sum_{a=1}^{\infty}|d_a|(a+1)ar_1^{a-1}+(1+r_1)\sum_{a=1}^{\infty}|d_a|ar_1^{a-1} \\ &+2r_1^2\sum_{a=2}^{\infty}|d_a(a-1)r_1^{a-2}<\infty \end{aligned}$$

This proves the theorem.

Theorem 2.6. *Suppose that α is a real parameter satisfying the condition $0 \leq \alpha \leq 1$ and $h : DR1 \to \mathbb{C}$ is analytic in $DR1$, $R1 > 1$ with $h(z) = \sum_{a=0}^{\infty} d_a z^a$*

1. *If f is not a polynomial of degree ≤ 1, then for all $1 \leq r1 < R1$, we have*

$$\|\tau_{m,\alpha}(h)-h\|_{r_1}\sim\frac{1}{m},\quad m\in\mathbb{N}$$

in Dr1 , where the constant in the equivalence depend on h and r1.

2. *If $1 \leq r_1 < r_2 < R_1$ and h is not a polynomial of degree $\leq \max\{1, q-1\}$ $(q \in N)$, we have*

$$\left\|\tau_{m,\alpha}^{(q)}(h)-h^{(q)}\right\|_{r1}\sim\frac{1}{m},\quad m\in\mathbb{N}$$

in Dr1 , where the constant in the equivalence depend on h, r1, r2 and q.

Proof. (1) Taking into account Theorem 2.5, there exist constants $0 < A1, A2 < \infty$ independent of m such that

$$A_1 \leq m\|\tau_{m,\alpha}(h)-h\|_{r_1} \leq A_2$$

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from which it readily follows that

$$\frac{A_1}{m} \leq \|\tau_{m,\alpha}(h) - h\|_{r_1} \leq \frac{A_2}{m}$$

in Dr1 . Therefore, we arrive at the desired result.

(2) Denoting by ζ the circle of radius $r2 > r1$ and center 0 (where $r2 > r1 \geq 1$), we have the inequality $|u - z| \geq r2 - r1$ valid for all $|z| \leq r1$ and $u \in \zeta$.

By the Cauchy's formula, it follows that for all $|z| \leq r1$ and $m, q \in N$

$$\tau_{m,\alpha}^{(q)}(h; z) - h^{(q)}(z) = \frac{q!}{2\pi\iota} \int_{\zeta} \frac{\tau_{m,\alpha}(h; u) - h(u)}{(u - z)^{q+1}} du$$

Thus, from Theorem 2.5 we get

$$\lim_{m\to\infty} m \left[\tau_{m,\alpha}^{(q)}(h; z) - h^{(q)}(z)\right] = \left[\frac{z(1-z)}{2} h''(z)\right]^{(q)}$$

uniformly in Dr1 . Therefore, there exist constants $0 < M1, M2 < \infty$ independent of m such that

$$M_1 \leq m \left\|\tau_{m,\alpha}^{(q)}(h) - h^{(q)}\right\|_{r_1} \leq M_2$$

which readily follows that

$$\frac{M_1}{m} \leq \left\|\tau_{m,\alpha}^{(q)}(h) - h^{(q)}\right\|_{r_1} \leq \frac{M_2}{m}$$

in 1 . This completes the proof.

Shape Preserving Properties of Complex α-Bernstein Operator:

In this section, we prove that beginning with an index, the complex α −Bernstein operators $\tau_{m,\alpha}(h; z)$ preserve some geometric properties of h such as starlikeness, convexity and spirallikeness in the unit disk.

Theorem 2.7*. Suppose that $H \subset \mathbb{C}$ is open such that $D1 \subset G$ and $h : H \to \mathbb{C}$ is analytic in H. Also let $\zeta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.*

1. *If h is univalent in $D1$, then there exists an index m_0 depending on h, such that for all $m \geq m0$, the complex α-Bernstein operators $\tau_{m,\alpha}(h; z)$ are univalent in $D1$.*
2. *If $h(0) = h'(0) - 1 = 0$ (and $h(z) \neq 0$, for all $z \in D1\{0\}$ in the case of spirallike of type ζ) and h is starlike (convex, spirallike of type ζ, respectively) in $D1$, that is for all $z \in D1$*

$$\text{Re}\left(\frac{zh'(z)}{h(z)}\right) > 0 \left(\text{Re}\left(\frac{zh''(z)}{h'(z)}\right) + 1 > 0,\ \text{Re}\left(e^{\iota\zeta}\frac{zh'(z)}{h(z)}\right) > 0, resp.\right),$$

then there exists an index m_0 depending on h (and on h and ζ for spirallikeness) such that for all $m \geq m_0$, the complex α-Bernstein operators $\tau_{m,\alpha}(h; z)$ are starlike (convex, spirallike of type ζ, respectively) in D1.

If $h(0) = h'(0) - 1 = 0$ (and $h(z) \neq 0$, for all $z \in D1\{0\}$ in the case of spirallike of type ζ) and h is starlike (convex, spirallike of type ζ, respectively) in $D1$, then for any disk of radius $0 < \sigma < 1$ and center 0 denoted by D_σ, there exists an index $m_0 = m_0(h, D_\sigma)$ (m_0 depends on ζ too in the case of spirallikeness), such that for all $m \geq m_0$, the complex α-Bernstein operators $\tau_{m,\alpha}(h; z)$ are starlike (convex, spirallike of type ζ, respectively) in D_σ, that is for all $z \in D_\sigma$.

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$$\operatorname{Re}\left(\frac{z\tau'_{m,\alpha}(h;z)}{\tau_{m,\alpha}(h;z)}\right) > 0$$

$$\left(\operatorname{Re}\left(\frac{z\tau''_{m,\alpha}(h;z)}{\tau_{m,\alpha}(h;z)}\right) + 1 > 0,\ \operatorname{Re}\left(e^{\iota\zeta}\frac{z\tau'_{m,\alpha}(h;z}{\tau_{m,\alpha}(h;z)}\right) > 0,\ resp.\right),$$

Proof. If h is univalent in $D1$, from the uniform convergence in Theorem 2:4 and the result concerning sequences of analytic functions converging locally uniformly to a univalent function, then it is immediate that for sufficiently large m, the complex α-Bernstein operators $\tau_{m,\alpha}(h;\ z)$ are univalent in $D1$ *(Using the theorem: [15] Suppose that Ω is a domain in $\mathbb{C}^n$ and that $h \in H(\Omega)$ is a biholomorphic mapping. Suppose $\{ha\}a \in N \subset H(\Omega)$ is a sequence of mappings such that $ha \to h$ locally uniformly on . Let A be a compact subset of Ω. Then there exists $a0 \geq 1$ such that $ha|A$ is injective for $a \geq a0$)*

Firstly, suppose that $h(0) = h'(0) - 1 = 0$ and h is starlike in $D1$. By Theorem 2.4 (2), we get that

$$\tau_{m,\alpha}(h;z) \to h(z),\quad \tau'_{m,\alpha}(h;z) \to h'(z)\ and\ \tau''_{m,\alpha}(h;z) \to h''(z)\ as\ m \to \infty$$

uniformly in $D1$. Now set

$$R_{m,\alpha}(h;z) = \frac{\tau_{m,\alpha}(h;z)}{(m-2+2\alpha)h\left(\frac{1}{m}\right) + (1-\alpha)h\left(\frac{2}{m}\right)} \qquad (34)$$

well defined for sufficiently large m. Taking into account $h(0) = h'(0) - 1 = 0$ and the univalence of h, we get

$$(m-2+2\alpha)h\left(\frac{1}{m}\right) + (1-\alpha)h\left(\frac{2}{m}\right) \neq 0$$

Now we will calculate the value of $Rm, \alpha;\_(h;\ z)$ at $z = 0$. So from equation (34)

$$R_{m,\alpha}(h;0) = \frac{\tau_{m,\alpha}(h;0)}{(m-2+2\alpha)h\left(\frac{1}{m}\right) + (1-\alpha)h\left(\frac{2}{m}\right)} \qquad (35)$$

Now we have to find the value of $\tau_{m,\alpha}(h;\ 0)$. As we know that

$$\tau_{m,\alpha}(h;z) = (1-\alpha)\sum_{t=0}^{m-1}\Gamma_t\binom{m-1}{t}z^t(1-z)^{m-t-1} + \alpha\sum_{t=0}^{m}h_t\binom{m}{t}z^t(1-z)^{m-t} \qquad (36)$$

Where

$$\Gamma_t = \left(1 - \frac{t}{m-1}\right)h_t + \frac{t}{m-1}h_{t+1},\quad m \geq 2$$

Put $z = 0$ in above equation we get

$$\tau_{m,\alpha}(h;0) = 0$$

Substitute the above value in equation (35)

$$R_{m,\alpha}(h;0) = \frac{\tau_{m,\alpha}(h;0)}{(m-2+2\alpha)\left(\frac{1}{m}\right) + (1-\alpha)h\left(\frac{2}{m}\right)} = 0$$

Differentiating equation (34) with respect to z and put $z = 0$, we get

$$R'_{m,\alpha}(h;0) = \frac{\tau'_{m,\alpha}(h;0)}{(m-2+2\alpha)h\left(\frac{1}{m}\right) + (1-\alpha)h\left(\frac{2}{m}\right)} \qquad (37)$$

So, we have to calculate the value of $\tau'_{m,\alpha}(h;\ z)$ at $z = 0$. Differentiating equation (36) with respect to z, we get

$$\begin{aligned}\tau'_{m,\alpha}(h;z) &= (1-\alpha)\sum_{t=0}^{m-1}\Gamma_t\binom{m-1}{t}\left[tz^{t-1}(1-z)^{m-t-1} - (m-t-1)z^t(1-z)^{m-t-2}\right] \\ &\quad +\alpha\sum_{t=0}^{m} h_t\binom{m}{t}\left[tz^{t-1}(1-z)^{m-t} - (m-t)z^t(1-z)^{m-t-1}\right]\end{aligned}$$

Where

$$\Gamma_t = \left(1-\frac{t}{m-1}\right)h_t + \frac{t}{m-1}h_{t+1} \quad m \geq 2$$

$$\begin{aligned}\tau'_{m,\alpha}(h;z) &= (1-\alpha)\sum_{t=0}^{m-1}\left(1-\frac{t}{m-1}\right)h_t + \frac{t}{m-1}h_{t+1}\binom{m-1}{t}\left[tz^{t-1}(1-z)^{m-t-1}\right. \\ &\quad \left.-(m-t-1)z^t(1-z)^{m-t-2}\right] \\ &\quad +\alpha\sum_{t=0}^{m} h_t\binom{m}{t}\left[tz^{t-1}(1-z)^{m-t} - (m-t)z^t(1-z)^{m-t-1}\right]\end{aligned}$$

When $t = 0, 2,3,4;$ ___ $, m - 1$ and $z = 0$ then we get

$$\tau'_{m,\alpha}(h;z) = 0$$

If $t = 1$ then

$$\begin{aligned}\tau'_{m,\alpha}(h;z) &= (1-\alpha)(m-1)\left(1-\frac{1}{m-1}\right)h_1 + \frac{1}{m-1}h_2\left[(1-z)^{m-2} - (m-2)z(1-z)^{m-3}\right] \\ &\quad +\alpha m h_1\left[(1-z)^{m-1} - (m-1)z(1-z)^{m-2}\right]\end{aligned}$$

If we $z = 0$ in above equation then we have

$$\tau'_{m,\alpha}(h;0) = (1-\alpha)(m-1)\left(1-\frac{1}{m-1}\right)h_1 + \frac{1}{m-1}h_2(1-0) + \alpha m h_1(1-0)$$

$$\tau'_{m,\alpha}(h;0) = (1-\alpha)(m-1)\left(1-\frac{1}{m-1}\right)h_1 + \frac{1}{m-1}h_2 + \alpha m h_1$$

Now, since $ht = h\left(\frac{t}{m}\right)$. So we have $h1 = h\left(\frac{1}{m}\right)$ and $h2 = h\left(\frac{2}{m}\right)$

Substituting these values in above equation

$$\tau'_{m,\alpha}(h;0) = (1-\alpha)(m-1)\left(1-\frac{1}{m-1}\right)h\left(\frac{1}{m}\right) + \frac{1}{m-1}h\left(\frac{2}{m}\right) + \alpha m h\left(\frac{1}{m}\right)$$

$$\tau'_{m,\alpha}(h;0) = (m-1-\alpha m+\alpha)\left[h\left(\frac{1}{m}\right) - \frac{1}{m-1}h\left(\frac{1}{m}\right) + \frac{1}{m-1}h\left(\frac{2}{m}\right)\right] + \alpha m h\left(\frac{1}{m}\right)$$

$$\begin{aligned}
\tau'_{m,\alpha}(h;0) &= mh\left(\frac{1}{m}\right) - \frac{m}{m-1}h\left(\frac{1}{m}\right) + \frac{m}{m-1}h\left(\frac{2}{m}\right) - h\left(\frac{1}{m}\right) + \frac{1}{m-1}h\left(\frac{1}{m}\right)\\
&\quad -\frac{1}{m-1}h\left(\frac{2}{m}\right) - \alpha mh\left(\frac{1}{m}\right) + \frac{\alpha m}{m-1}h\left(\frac{1}{m}\right) - \frac{\alpha m}{m-1}h\left(\frac{2}{m}\right) + \alpha h\left(\frac{1}{m}\right)\\
&\quad -\frac{\alpha}{m-1}h\left(\frac{1}{m}\right) + \frac{\alpha}{m-1}h\left(\frac{2}{m}\right) + \alpha mh\left(\frac{1}{m}\right)\\
\tau'_{m,\alpha}(h;0) &= mh\left(\frac{1}{m}\right) - \frac{m}{m-1}h\left(\frac{1}{m}\right) + \frac{m}{m-1}h\left(\frac{2}{m}\right) - h\left(\frac{1}{m}\right) + \frac{1}{m-1}h\left(\frac{1}{m}\right)\\
&\quad -\frac{1}{m-1}h\left(\frac{2}{m}\right) + \frac{\alpha m}{m-1}h\left(\frac{1}{m}\right) - \frac{\alpha m}{m-1}h\left(\frac{2}{m}\right) + \alpha h\left(\frac{1}{m}\right) - \frac{\alpha}{m-1}h\left(\frac{1}{m}\right)\\
&\quad +\frac{\alpha}{m-1}h\left(\frac{2}{m}\right)\\
\tau'_{m,\alpha}(h;0) &= \left(m - \frac{m}{m-1} - 1 + \frac{1}{m-1} + \frac{\alpha m}{m-1} + \alpha - \frac{\alpha}{m-1}\right)h\left(\frac{1}{m}\right)\\
&\quad + \left(\frac{m}{m-1} - \frac{1}{m-1} - \frac{\alpha m}{m-1} + \frac{\alpha}{m-1}\right)m\left(\frac{2}{m}\right)\\
\tau'_{m,\alpha}(h;0) &= \left(m - 1 + \alpha - \frac{m-1}{m-1} + \alpha\frac{m-1}{m-1}\right)h\left(\frac{1}{m}\right) + \left(\frac{m-1}{m-1} - \alpha\frac{m-1}{m-1}\right)h\left(\frac{2}{m}\right)\\
\tau'_{m,\alpha}(h;0) &= (m-1+\alpha-1+\alpha)\,h\left(\frac{1}{m}\right) + (1-\alpha)\,h\left(\frac{2}{m}\right)\\
\tau'_{m,\alpha}(h;0) &= (m-2+2\alpha)\,h\left(\frac{1}{m}\right) + (1-\alpha)\,h\left(\frac{2}{m}\right)
\end{aligned}$$

Substituting the above value in equation (37), we get

$$\begin{aligned}
R'_{m,\alpha}(h;0) &= \frac{(m-2+2\alpha)\,h\left(\frac{1}{m}\right) + (1-\alpha)\,h\left(\frac{2}{m}\right)}{(m-2+2\alpha)\,h\left(\frac{1}{m}\right) + (1-\alpha)\,h\left(\frac{2}{m}\right)}\\
R'_{m,\alpha}(h;0) &= 1
\end{aligned}$$

And

$$(m-2+2\alpha)h\left(\frac{1}{m}\right) + (1-\alpha)h\left(\frac{2}{m}\right)$$

$$= \frac{h\left(\frac{1}{m}\right) - h(0)}{\frac{1}{m}}\frac{(m-2+2\alpha)}{m} + \frac{h\left(\frac{2}{m}\right) - h(0)}{\frac{2}{m}}\frac{2(1-\alpha)}{m} \to h'(0) = 1$$

as $m \to \infty$, we obtain

$$R_{m,\alpha}(h;z) \to h(z), \quad R'_{m,\alpha}(h;z) \to h'(z)$$

And

$$R''_{m,\alpha}(h;z) \to h''(z)$$

uniformly in $D1$.

From the hypothesis we obtain $|h(z)| > 0$ for all $z \in D1$ with $z \neq 0$, from the univalance of h in $D1$, implies that we can write $h(z) = z\Gamma(z)$, with $\Gamma(z) \neq 0$, for all $z \in D1$, where Γ is analytic in $D1$ and continuous in $D1$.

Writing $Rm,\alpha(h;\ z)$ in the form $Rm,\alpha(h;\ z) = zSm,\alpha(h;\ z)$, obviously $Sm,\alpha(h;\ z)$ is a polynomial of degree $\leq m - 1$. Let $|z| = 1$. Then we get

$$\left|h(z) - R_{m,\alpha}(h;z)\right| = |z|\left|\Gamma(z) - S_{m,\alpha}(h;z)\right|$$
$$\left|h(z) - R_{m,\alpha}(h;z)\right| = \left|\Gamma(z) - S_{m,\alpha}(h;z)\right|$$

which gives the uniform convergence of $Sm,\alpha(h)$ to Γ' by the uniform convergence of $R'n,\alpha(h)$ to h' and of $Sm,\alpha(h)$ to Γ, with the help of maximum modulus principle. So for $|z| = 1$, we have

$$\frac{zR'_{m,\alpha}(h;z)}{R_{m,\alpha}(h;z)} = \frac{z\left[zS'_{m,\alpha}(h;z) + S_{m,\alpha}(h;z)\right]}{zS_{m,\alpha}(h;z)}$$
$$\frac{zR'_{m,\alpha}(h;z)}{R_{m,\alpha}(h;z)} = \frac{zS'_{m,\alpha}(h;z) + S_{m,\alpha}(h;z)}{S_{m,\alpha}(h;z)} \to \frac{z\Gamma'(z) + \Gamma(z)}{\Gamma(z)} = \frac{h'(z)}{\Gamma(z)}$$
$$\frac{zR'_{m,\alpha}(h;z)}{R_{m,\alpha}(h;z)} = \frac{zh'(z)}{h(z)}$$

which again by the maximum modulus principle, follows

$$\frac{zR'_{m,\alpha}(h;z)}{R_{m,\alpha}(h;z)} \to \frac{zh'(z)}{h(z)}$$

as $m \to \infty$; uniformly in $D1$.

Since $Re\left(\frac{zh'(z)}{h(z)}\right)$ is continuous in $D1$, there exists $\eta \in (0,1)$ such that

$$\operatorname{Re}\left(\frac{zh'(z)}{h(z)}\right) \geq \eta, \ for\ all\ z \in \overline{D_1}.$$

Thus

$$\operatorname{Re}\left(\frac{zR'_{m,\alpha}(h;z)}{R_{m,\alpha}(h;z)}\right) \to \operatorname{Re}\left(\frac{zh'(z)}{h(z)}\right) \geq \eta > 0$$

uniformly in $D1$, i.e. for any $0 < \upsilon < \eta$, there is m_0 such that for all $m \geq m_0$, we get

$$\operatorname{Re}\left(\frac{zR'_{m,\alpha}(h;z)}{R_{m,\alpha}(h;z)}\right) > \nu > 0, \ for\ all\ z \in \overline{D_1}$$

Since

$$R_{m,\alpha}(h;z) = \frac{\tau_{m,\alpha}(h;z)}{(m-2+2\alpha)h\left(\frac{1}{m}\right) + (1-\alpha)h\left(\frac{2}{m}\right)}$$

then $Rm,\alpha(h;\ z)$ differs from $\tau_{m,\alpha}(h;\ z)$ only be a constant. This Proves the starlikeness of $\tau_{m,\alpha}(h;\ z)$ for sufficiently large m in $D1$.

If h is starlike only in $D1$, the proof is identical with the first part. The only difference is that we reason for D_σ with $0 < \sigma < 1$ instead of $D1$.

The proofs in the case when h is convex or spirallike of order ζ are similar and follow from the following uniform convergence (on $D1$ or on D_σ with $0 < \sigma < 1$) as $m \to \infty$

$$\mathrm{Re}\left(\frac{zR'_{m,\alpha}(h;z)}{R_{m,\alpha}(h;z)}\right)+1 \to \mathrm{Re}\frac{zh''(z)}{h'(z)}+1$$

And

$$\mathrm{Re}\left(e^{\iota\zeta}\frac{zR'_{m,\alpha}(h;z)}{R_{m,\alpha}(h;z)}\right) \to \mathrm{Re}\left(e^{\iota\zeta}\frac{zh'(z)}{h(z)}\right)$$

Thus, the proof is complete.

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