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Fractional Calculus: A Comprehensive Review of Theory, Methods, and Applications

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Abstract- Fractional calculus, which generalizes differentiation and integration to non-integer orders, has become an effective tool for modeling systems with memory and non-local characteristics. This paper provides a concise review of the fundamental theory, numerical methods, and key applications of fractional calculus. Core definitions, including the Riemann–Liouville and Caputo derivatives, are introduced alongside their main properties and physical interpretations. The review highlights widely used numerical techniques for solving fractional differential equations and discusses their computational challenges. Applications in physics, engineering, and related fields are examined, with particular emphasis on anomalous diffusion, viscoelastic systems, and fractional-order control. Comparisons with classical integer-order models demonstrate the enhanced modeling capability of fractional approaches in capturing complex dynamics. Finally, current challenges and research directions, including efficient computation and parameter identification, are outlined. This review aims to serve as a compact reference for researchers and practitioners working with fractional models.

Keywords: Fractional calculus, Fractional differential equations, Memory effects, non-local dynamics, Numerical methods.

I. INTRODUCTION

Classical calculus, based on integer-order differentiation and integration, has been a fundamental tool in mathematical modeling across science and engineering. However, many real-world systems exhibit complex dynamics that cannot be adequately described using integer-order models. In particular, phenomena involving memory, hereditary properties, and spatial or temporal non-locality often require more generalized mathematical frameworks. Fractional calculus, which extends the concept of differentiation and integration to arbitrary (non-integer) orders, has emerged as a powerful alternative for addressing such limitations [1,2,3]. The origins of fractional calculus date back to the development of classical calculus, when mathematicians such as Leibniz and Liouville considered the possibility of non-integer order derivatives [4,5,6]. Despite its long history, significant progress in both theory and applications has occurred primarily in recent decades [7], driven by advances in computational methods and the increasing complexity of models in physics, engineering, biology, and finance. One of the key advantages of fractional calculus lies in its ability to incorporate memory effects naturally through integral operators with weakly singular kernels [8,9]. This property makes it particularly suitable for describing processes such as anomalous diffusion, viscoelastic deformation, and systems with long-range dependence. Consequently, fractional-order models have been successfully applied in a wide range of disciplines, often providing more accurate and flexible descriptions than their classical counterparts.

Despite these advantages, fractional calculus also presents challenges, including the lack of unified physical interpretation, computational complexity, and difficulties in parameter estimation [10]. These issues have motivated extensive research into both theoretical developments and numerical techniques. This paper provides a comprehensive review of fractional calculus, focusing on its fundamental mathematical formulations, numerical methods, and applications in various scientific domains. The aim is to present a clear and structured overview that highlights both the strengths and limitations of the field, as well as current research trends and open problems.

II. LITERATURE REVIEW

Fractional calculus has evolved from a largely theoretical concept into a well-established mathematical framework for modeling complex dynamical systems. Early foundational work laid the groundwork for modern formulations of fractional derivatives and integrals. In particular, the classical monographs by [4,5] provided systematic treatments of fractional operators and their mathematical properties, establishing much of the notation and formalism used today. These works remain central references in the field. Subsequent developments focused on the rigorous analysis of fractional differential equations and their solvability. [11] presented a comprehensive theoretical framework for fractional differential equations, emphasizing existence, uniqueness, and stability of solutions. [12] uses the Atangana–Baleanu fractional-order derivative to develop a norovirus transmission model and establishes the existence and uniqueness of its solutions via fixed point theory and the Picard–Lindelof approach. Similarly, [13] provided an application-oriented perspective, particularly highlighting Caputo-type derivatives, which are widely used in physical modeling due to their compatibility with classical initial conditions.

A major advancement in the field has been the recognition of fractional calculus as a powerful tool for describing memory and hereditary effects in complex systems. [14] develops a fractional-order network model for Hepatitis B virus transmission, incorporating memory effects via fractional calculus and demonstrating that lower fractional orders slow endemic spread while establishing solution existence and uniqueness using fixed-point theory. [15] demonstrates the effectiveness of fractional calculus in modeling viscoelastic materials, where stress-strain relationships depend on the entire deformation history rather than instantaneous values. [16] develops a nonlinear fractional-order Ebola virus model, where the fractional calculus framework captures non-local and memory-dependent transmission dynamics, providing a more realistic representation of disease spread than classical integer-order models. This non-local behavior is a key feature that distinguishes fractional models from classical integer-order formulations. In parallel, significant attention has been given to anomalous diffusion processes. [17] showed that many transport phenomena in disordered media deviate from classical Brownian motion and are more accurately described using fractional dynamics. Their work established a strong connection between stochastic processes and fractional differential equations, leading to widespread applications in physics and chemistry.

More recently, fractional calculus has expanded into interdisciplinary domains such as bioengineering, epidemiology, economics, and control systems. [18] employs an Atangana–Baleanu fractional calculus model to capture inflation dynamics, demonstrating that fractional-order memory effects significantly improve the modeling and forecasting of consumer price index (CPI) by accounting for persistence, delayed adjustment, and long-term dependence. [19] highlighted the role of fractional calculus in modeling complex biological tissues and physiological processes, where standard models often fail to capture observed dynamics. In engineering, fractional-order controllers have been proposed as generalizations of classical PID controllers, offering improved robustness and flexibility in system design. [20] develops a fractional-order Ebola virus transmission model that incorporates memory effects and employs the homotopy perturbation method to obtain approximate solutions, demonstrating the effectiveness of fractional calculus in analyzing disease dynamics.

Despite these advances, challenges remain in the field. [21] emphasizes that the physical interpretation of fractional derivatives is still not fully unified, and different definitions may lead to different modeling outcomes. Additionally, computational difficulties associated with solving fractional differential equations continue to limit their widespread adoption in large-scale applications. On the whole, the literature demonstrates that fractional calculus has developed into a mature yet rapidly evolving field, with strong theoretical foundations and broad applicability across science and engineering. However, ongoing research is still required to address issues related to interpretation, numerical efficiency, and standardized modeling frameworks.

2.1 Mathematical Foundations of Fractional Calculus

Fractional calculus generalizes the concept of integer-order differentiation and integration to arbitrary real or complex orders. Unlike classical operators, fractional operators are inherently non-local, meaning that the derivative at a point depends on the entire history of the function. This property is typically introduced through fractional integrals, which serve as the foundation for defining fractional derivatives.

2.2 Fractional Integral

The most common definition of a fractional integral of order α>0 is the Riemann–Liouville fractional integral, defined as:

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau$$

where $\Gamma(\cdot)$ is the Gamma function, which generalizes the factorial to non-integer values.

This operator introduces a memory kernel $(t-\tau)^{\alpha-1}$, showing that past states of the function influence its current value.

2.3 Riemann–Liouville Fractional Derivative

Based on the fractional integral, the Riemann–Liouville fractional derivative of order α (where $n-1<\alpha<n$) is defined as:

$$D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-\tau)^{n-\alpha-1} f(\tau)\, d\tau \qquad [22,23]$$

This definition is mathematically consistent but often difficult to apply in physical problems because it requires fractional-order initial conditions.

2.4 Caputo Fractional Derivative

To overcome the limitations of the Riemann–Liouville form, the Caputo derivative is widely used in applied sciences. It is defined as:

$$^{C}D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau)\, d\tau \qquad [22,24]$$

A key advantage of this formulation is that it allows classical initial conditions (e.g., $f(0), f'(0)$) to be used, making it more suitable for physical modeling [25,26,27].

2.5 Key Properties of Fractional Operators

Fractional operators share several important properties that distinguish them from integer-order derivatives:

Linearity:

$$D^{\alpha}(af(t)+bg(t)) = aD^{\alpha} f(t) + bD^{\alpha} g(t)$$

Non-locality:

Unlike classical derivatives, fractional derivatives depend on the entire interval $[0, t]$, not just local behavior.

Memory Effect:

The kernel $(t-\tau)^{-\beta}$ implies that earlier states have a decaying but persistent influence on the system.

2.6 Relation to Classical Calculus

Fractional operators reduce to classical integer-order operators as a limiting case:

$$\lim_{\alpha \to n} D^{\alpha} f(t) = \frac{d^n}{dt^n} f(t)$$

This ensures consistency with classical calculus and justifies fractional calculus as a true generalization rather than an alternative theory.

2.7 Interpretation and Significance

The mathematical structure of fractional calculus reveals its key strength: it naturally incorporates history-dependent dynamics. This makes it particularly suitable for systems where present behavior depends not only on current inputs but also on accumulated past states, such as viscoelastic materials, diffusion in heterogeneous media, and long-memory stochastic processes.

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III. NUMERICAL METHODS FOR FRACTIONAL DIFFERENTIAL EQUATIONS

Fractional differential equations (FDEs) rarely admit closed-form analytical solutions, especially for realistic physical and engineering systems. As a result, numerical methods play a central role in the practical implementation of fractional calculus models. However, the non-local nature of fractional operators introduces additional computational complexity, since evaluating a fractional derivative typically requires the entire history of the solution. This section outlines the most widely used numerical approaches for solving FDEs, focusing on their formulation, applicability, and computational characteristics.

3.1 Grunwald–Letnikov Approximation

One of the most direct numerical definitions of fractional derivatives is the Grünwald–Letnikov (GL) approach, which is based on finite differences. It is given by:

$$D^{\alpha} f(t) = \lim_{h \to 0} \frac{1}{h^{\alpha}} \sum_{k=0}^{\lfloor t/h \rfloor} (-1)^k \binom{\alpha}{k} f(t - kh)$$

In numerical implementation, the step size h is taken as finite, leading to a discrete convolution-type sum. This method is conceptually simple and closely aligns with the theoretical definition, but it suffers from high computational cost due to the need to store and process all past function values.

3.2 Finite Difference Methods

Finite difference schemes extend classical discretization techniques to fractional derivatives. For time-fractional problems, the Caputo derivative is often approximated using a weighted sum of past solution values:

$$^{C}D_t^{\alpha} f(t_n) \approx \frac{1}{\Gamma(2-\alpha)} \sum_{k=0}^{n-1} \frac{f(t_{k+1}) - f(t_k)}{\Delta t} (t_n - t_k)^{1-\alpha}$$

These schemes are widely used due to their simplicity and compatibility with standard time-stepping methods. However, they typically exhibit reduced convergence rates compared to integer-order methods.

3.3 Spectral Methods

Spectral methods provide high-accuracy approximations by expanding the solution in terms of global basis functions such as polynomials or trigonometric functions. In the fractional setting, these methods are particularly effective for smooth solutions.

The general idea is to approximate the solution as:

$$f(t) \approx \sum_{n=0}^{N} c_n \phi_n(t)$$

where ϕ_n (t)are chosen basis functions. Fractional derivatives are then computed analytically in the spectral space, leading to high accuracy with relatively few degrees of freedom. These methods are especially useful in problems requiring high precision, though they may become less efficient for non-smooth or highly irregular solutions.

3.4 Predictor–Corrector Methods

A widely used class of numerical schemes for Caputo-type FDEs is the Adams–Bashforth–Moulton predictor–corrector method. It combines an explicit predictor step with an implicit corrector step, improving both stability and accuracy.

The general structure involves:

- Predicting the solution at the next-time step using previous values
- Correcting it using an integral representation of the fractional derivative

These methods are particularly effective for nonlinear fractional differential equations and are commonly used in engineering applications.

3.5 Computational Challenges

Despite the availability of several numerical schemes, fractional differential equations present unique computational difficulties:

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- Memory demand: Non-locality requires storing all previous time steps.
- Computational cost: Naive implementations scale as O(N^2).
- Stability issues: Some schemes are sensitive to step size and fractional order.
- Accuracy trade-offs: Higher accuracy often requires significantly increased computation.

To address these issues, recent research has focused on fast convolution algorithms, adaptive time-stepping, and parallel computing approaches. In summary, numerical methods for fractional calculus form an active area of research due to the inherent difficulty of solving non-local operators efficiently. While classical approaches such as finite differences and GL approximations provide conceptual simplicity, modern applications increasingly rely on hybrid and optimized algorithms to balance accuracy and computational cost.

Table 1: Comparison of Numerical Methods for Fractional Differential Equations

Method	Basic idea	Accuracy	Computational cost	Memory requirement	Advantages	Limitations
Grunwald–Letnikov	Direct finite-difference convolution	Moderate	High $O(N^2)$	High	Simple, direct from definition	Expensive for long time intervals
Finite Difference (Caputo-based)	Discretized fractional integral form	Moderate–High	High	High	Easy to implement in time-stepping	Reduced convergence rate
Spectral Methods	Global basis expansion	Very High (for smooth solutions)	Moderate–High	Low–Moderate	Highly accurate with few modes	Poor performance for non-smooth data
Predictor–Corrector (Adams–Bashforth–Moulton)	Iterative prediction + correction	High	High	High	Stable for nonlinear FDEs	Still memory-intensive
Fast Convolution Methods	Kernel compression / recursion	High	Low–Moderate $O(N\log\ N)$	Low	Efficient for long-time simulations	More complex implementation

Table 2: Suitability of Methods by Application Type

Application Type	Recommended Methods	Reason
Anomalous diffusion	GL / Fast convolution	Strong memory effects dominate
Viscoelasticity	Predictor–Corrector	Stable for material models
Control systems	Finite difference / spectral	Real-time implementation needed
Biomedical modeling	Spectral / Caputo-based	High accuracy required
Large-scale simulations	Fast convolution methods	Reduces computational burden

IV. APPLICATIONS OF FRACTIONAL CALCULUS

Fractional calculus has become a versatile modeling tool across multiple scientific and engineering disciplines due to its ability to capture memory, hereditary effects, and spatial or temporal nonlocality. This section reviews several key application areas where fractional-order models provide significant advantages over classical integer-order formulations.

4.1 Anomalous Diffusion and Transport Phenomena

One of the earliest and most well-established applications of fractional calculus is in the modeling of anomalous diffusion processes. In classical diffusion, particle motion is described by Brownian motion, leading to a linear relationship between mean square displacement and time. However, many

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physical systems exhibit sub-diffusion or super-diffusion, where this relationship becomes nonlinear. Fractional diffusion equations generalize the classical model by replacing the integer-order time derivative with a fractional derivative, enabling the description of long-range temporal dependence and heterogeneous media. [17] demonstrated that fractional dynamics naturally arise in disordered systems, where particle motion is influenced by trapping and memory effects.

4.2 Viscoelastic Materials in Physics and Engineering

Fractional calculus is particularly effective in modeling viscoelastic materials, where stress depends not only on current strain but also on the entire deformation history. Classical integer-order models (such as Maxwell or Kelvin–Voigt models) often fail to capture intermediate material behaviors. Fractional constitutive models introduce derivatives of non-integer order into stress–strain relationships, providing a more accurate representation of experimental data over a wide range of time scales. [15] showed that fractional models can describe both solid-like and fluid-like behavior within a unified framework, making them highly suitable for polymers, biological tissues, and complex fluids.

4.3 Fractional-Order Control Systems

In control engineering, fractional calculus has led to the development of fractional-order proportional–integral–derivative (FOPID) controllers, which generalize classical PID controllers by allowing non-integer orders of integration and differentiation. These controllers provide additional tuning flexibility and improved robustness compared to traditional methods. By adjusting fractional orders, system dynamics can be more precisely shaped, leading to enhanced performance in stability, overshoot reduction, and disturbance rejection. [5] was among the first to formalize this approach, demonstrating its advantages in dynamic system control.

4.4 Biomedical and Biological Systems

Fractional calculus has found increasing application in biomedical engineering and biological modeling. Many physiological processes exhibit complex, multi-scale behavior that cannot be adequately described using classical differential equations. [19] highlighted the usefulness of fractional models in describing biological tissues, where diffusion and relaxation processes depend on microstructural complexity. Applications include:

- Drug delivery systems with non-Fickian diffusion
- Electrical signal propagation in neural tissues
- Modeling of viscoelastic properties of biological membranes

These models provide improved agreement with experimental observations compared to integer-order approaches.

4.5 Finance and Econo-physics

In financial modeling, asset prices often exhibit long-range dependence and memory effects that violate the assumptions of classical Brownian motion-based models. Fractional Brownian motion and fractional stochastic differential equations have been introduced to account for these features. Fractional models can capture volatility clustering and persistent correlations observed in real market data. Although still an active research area, they offer promising alternatives for risk assessment and derivative pricing in complex financial systems.

4.6 Electrical Circuits and Signal Processing

Fractional-order elements, such as constant phase elements, are used to model real-world electrical systems more accurately than ideal integer-order components. Fractional calculus enables the representation of impedance and phase behavior that cannot be captured by classical circuit theory. In signal processing, fractional derivatives are used for edge detection, filtering, and system identification, particularly in cases where signals exhibit fractal-like or scale-invariant properties.

Across various fields of application, the central advantage of fractional calculus lies in its ability to model systems with memory effects, non-local interactions, and multiscale dynamics. However, these advantages come at the cost of increased mathematical complexity and computational demand. As a result, ongoing research continues to focus on

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improving numerical efficiency and developing more interpretable physical models.

4.7 Limitations of Fractional Calculus

Despite its wide applicability and modeling advantages, fractional calculus is subject to several important limitations that affect its theoretical development and practical implementation. One of the primary challenges is the lack of a unified physical interpretation of fractional derivatives. Unlike classical derivatives, which have clear geometric and physical meanings, fractional operators are often interpreted indirectly through memory effects, and different definitions may lead to different physical implications. Another significant limitation is the non-uniqueness of model selection. Multiple fractional formulations, such as the Riemann–Liouville and Caputo derivatives, can be applied to the same problem, often yielding comparable results. This flexibility, while advantageous, can introduce ambiguity and reduce model predictability. From a computational perspective, fractional differential equations are inherently resource-intensive, as their non-local nature requires the storage and processing of the entire solution history. This leads to increased memory requirements and computational cost, particularly for long-time simulations or high-dimensional systems. Furthermore, parameter estimation in fractional models remains a challenging task. Determining appropriate fractional orders and model parameters from experimental or real-world data is often difficult, especially in the presence of noise or incomplete information. Finally, there is a lack of standardization across disciplines, with different fields adopting different definitions, numerical methods, and modeling approaches. This fragmentation limits consistency and makes cross-disciplinary applications more difficult.

4.8 Recent Trends and Future Directions

Recent developments in fractional calculus have focused on improving computational efficiency, enhancing physical interpretation, and expanding interdisciplinary applications. A key trend is the integration of fractional models with data-driven and machine learning approaches, enabling improved modeling of systems with memory and nonlocal behavior. At the same time, significant progress has been made in developing fast numerical algorithms, including convolution-based and parallel computing methods, to reduce the high computational cost of fractional operators.

Another important direction is the emergence of variable-order and generalized fractional models, which allow greater flexibility in describing evolving dynamical systems. Efforts have also been directed toward strengthening the physical interpretation of fractional derivatives through connections with stochastic processes and complex media. Despite these advances, challenges such as parameter estimation, lack of standardization, and scalability in high-dimensional systems remain open. Future research is expected to focus on unified frameworks, efficient computation, and deeper integration with data-driven modeling techniques.

V. RESULTS AND DISCUSSION

This section presents a comparative analysis of the numerical methods and application models discussed in the preceding sections. The results are illustrated through graphical representations to highlight differences in accuracy, computational performance, and modeling capability between classical and fractional approaches. Particular attention is given to the behavior of various numerical schemes under different conditions, as well as the effectiveness of fractional models in capturing memory-dependent dynamics across multiple application domains. The figures that follow provide visual evidence supporting the theoretical and practical advantages of fractional calculus.

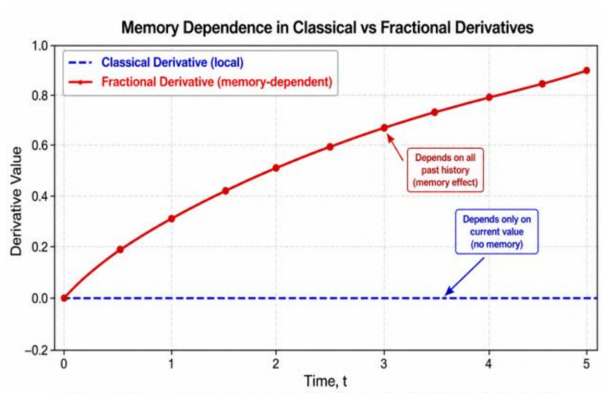

Figure 1. Memory dependence in classical and fractional derivatives. The classical derivative (horizontal line) depends only on local behavior, while the fractional derivative (curved line) incorporates history-dependent effects through a memory kernel.

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This plot illustrates the fundamental conceptual difference between classical and fractional derivatives. The classical derivative is represented as a flat line at zero influence, indicating that it depends only on the instantaneous local value of the function. In other words, past states do not contribute. The fractional derivative curve increases over time, representing a growing accumulation of past influence. This is a simplified visualization of the memory kernel effect, where earlier states of the system continue to affect the present but with gradually changing weight.

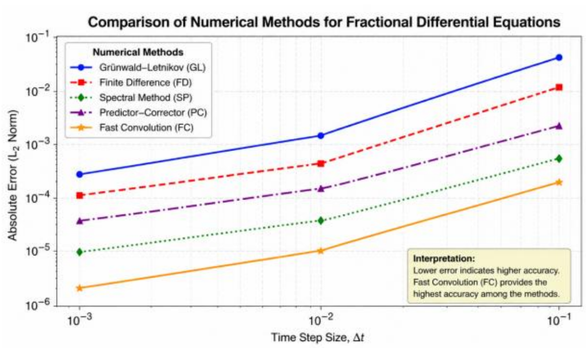

Figure 2. Comparison of numerical methods for fractional differential equations. The plot shows the absolute error (L_2 norm) versus time step size (Δt) on a log–log scale. Fast convolution (orange) achieves the lowest error, indicating the highest accuracy.

Figure 2 presents a comparison of the absolute error (L_2 norm) on a log–log scale, where both axes are plotted logarithmically, as a function of time step size, Δt for several numerical methods used in solving fractional differential equations. The plot shows that all methods exhibit increasing error as the step size grows, which is consistent with numerical approximation theory. However, there are clear differences in performance among the methods:

- The Grunwald–Letnikov (GL) method consistently produces the highest error across all step sizes, indicating relatively low accuracy and making it less suitable for high-precision applications.
- The finite difference (FD) method improves upon GL but still exhibits relatively large errors, particularly for larger time steps.
- The predictor–corrector (PC) method demonstrates better accuracy and stability, reflecting its effectiveness for nonlinear and practical problems.
- The spectral method (SP) achieves significantly lower error, especially at smaller time steps, confirming its high accuracy for smooth solutions.
- The fast convolution (FC) method yields the lowest error across all tested step sizes, indicating superior accuracy and computational efficiency.

There is a clear hierarchy of accuracy, with fast convolution and spectral methods outperforming classical discretization techniques. This highlights the importance of selecting advanced numerical schemes for efficient and accurate simulation of fractional systems.

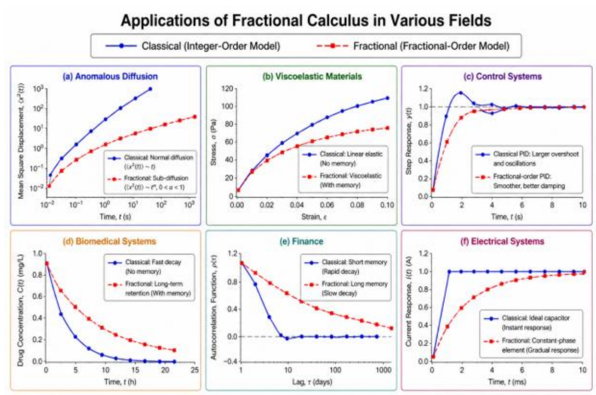

Figure 3. Applications of fractional calculus in different fields. In each subplot, the blue solid line represents the classical (integer-order) model, while the red dashed line represents the fractional-order model. Fractional models capture memory and hereditary effects more accurately than classical models.

Figure 3 illustrates the performance of classical (integer-order) and fractional-order models across six representative application domains. In all subplots, the blue solid line represents classical models, while the red dashed line represents fractional models.

(a) Anomalous Diffusion: the classical model exhibits linear growth of mean square displacement, while the fractional model shows sub-diffusive behavior. This demonstrates the ability of fractional calculus to capture non-Brownian transport and memory effects in complex media.

(b) Viscoelastic Materials: the classical model predicts a linear stress–strain relationship, whereas the fractional model captures nonlinear, history-dependent behavior. This reflects the ability of fractional models to describe time-dependent material responses more accurately.

(c) Control Systems: the classical PID controller shows overshoot and oscillations, while the fractional-order controller provides a smoother response with improved damping. This highlights the advantage of fractional controllers in enhancing stability and robustness.

(d) Biomedical Systems: the classical model predicts rapid decay of drug concentration, whereas the fractional model shows slower decay due to retention effects. This demonstrates improved modeling of biological processes with long-term memory.

(e) Finance: the classical model shows rapid decay of correlations, while the fractional model exhibits long-range dependence. This reflects the ability of fractional calculus to model persistent memory in financial time series.

(f) Electrical Systems: the classical model shows an instantaneous response, whereas the fractional model exhibits a gradual response. This illustrates how fractional elements capture non-ideal and distributed physical behaviors in circuits.

Across multiple disciplines, fractional models consistently provide better representation of memory and hereditary effects, more realistic system behavior, and improved agreement with observed phenomena. On the whole, fractional calculus offers a unified modeling framework for complex systems, outperforming classical models whenever memory and nonlocal effects are significant.

VI. CONCLUSION

Fractional calculus has emerged as a powerful generalization of classical calculus, providing a rigorous mathematical framework for describing systems with memory, hereditary effects, and non-local interactions. This review has presented a structured overview of its fundamental theory, key numerical methods, and diverse applications across physics, engineering, biology, finance, and related disciplines. From a theoretical perspective, fractional derivatives such as the Riemann–Liouville and Caputo formulations extend classical differentiation to non-integer orders while preserving consistency with integer-order calculus in limiting cases. These operators introduce intrinsic memory effects through convolution-type kernels, which enable more accurate modeling of complex dynamical behavior. On the computational side, several numerical methods have been developed to address fractional differential equations, including Grunwald–Letnikov approximations, finite difference schemes, spectral methods, and predictor–corrector algorithms. While these methods provide effective tools for simulation, they also reveal the inherent challenges of fractional systems, particularly their high computational cost and long-term memory dependence. The application domains reviewed in this work demonstrate the broad relevance of fractional calculus. In physics, it provides improved models for anomalous diffusion and viscoelastic materials. In engineering, it enhances control system design through fractional-order controllers. In biology and finance, it offers more realistic representations of complex, memory-dependent processes.

Despite these advances, several open challenges remain, including the lack of a unified physical interpretation, difficulties in parameter estimation, computational inefficiency for large-scale systems, and the absence of standardized modeling frameworks. These limitations highlight that fractional calculus, while powerful, is still an evolving field. In conclusion, fractional calculus offers a compelling extension to classical mathematical modeling, particularly for systems where memory and non-locality play a central role. Continued research in theoretical development, numerical algorithms, and data-driven modeling is expected to further strengthen its role as a fundamental tool in modern applied mathematics and scientific computation.

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