Formulation of the Governing Partial Differential Equation for Horizontal Velocity as a Function of Vertical Depth with Depth-Dependent Hydraulic Conductivity in Confined Aquifer Systems

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Abstract- In confined aquifer systems, hydraulic conductivity varies with depth due to compaction and lithological changes, but the partial differential equations governing horizontal groundwater velocity under such conditions have not been expressed in a form amenable to systematic analytical treatment. This paper formulates the governing PDEs for two commonly used conductivity-depth relationships: the exponential model $K(z)=K_0 e^{-\alpha z}$ and the power-law model $K(z)=K_0 z^{-\beta}$. By inserting each model into the mass conservation equation coupled with Darcy's law and carrying out the required differentiation, both equations are reduced to the common form $h_{xx}+h_{zz}-f(z) h_z=g(z) h_t$. For exponential conductivity, $f(z)=\alpha$ is constant and $g(z)=(S_s/K_0) e^{\alpha z}$ grows without bound; for power-law conductivity, $f(z)=\beta/z$ is singular at the origin and $g(z)=(S_s/K_0) z^{\beta}$ grows algebraically. Setting the time derivative to zero produces two elliptic equations: one with constant coefficients (exponential) and one retaining the β/z singularity (power-law). A term-by-term comparison reveals three structural contrasts—in the convective coefficient, the diffusivity growth rate, and the steady-state coefficient type—that govern the applicability of different analytical solution methods. Recovery of the horizontal velocity from the head field via Darcy's law is stated explicitly.

Keywords: Confined Aquifer, Depth-Dependent Hydraulic Conductivity, Governing PDE, Exponential Decay; Power-Law Decay, Darcy's Law, Velocity-Depth Relationship

I. INTRODUCTION

Overburden pressure in confined aquifer systems compresses the geological matrix progressively with depth, reducing pore connectivity and lowering hydraulic conductivity. Field observations and laboratory measurements indicate that this reduction follows either an exponential law $K(z)=K_0 e^{-\alpha z}$ or a power-law $K(z)=K_0 z^{-\beta}$, depending on lithological composition [5, 13] . Because horizontal groundwater velocity at any depth is proportional to the local conductivity through Darcy's law, the depth-variation of K produces a velocity profile that differs markedly from the uniform prediction [1, 3] .

Early treatments of confined aquifer flow [3] took K to be spatially uniform, reducing the governing equation to a standard diffusion problem. Later work relaxed this assumption. Rotzoll and El-Kadi [12] embedded exponential K(z) in a finite-difference simulator for the Pearl Harbor Aquifer, Hawai'i, and obtained improved velocity distributions relative to uniform-K runs. Jiang et al. [5] showed by inverse modelling that depth-varying conductivity is necessary to reproduce observed heads in layered formations. Patel et al. [11] introduced a power-law conductivity function for inclined recharge problems and solved the resulting equation numerically. Jiang et al. [6] constructed Fourier-based analytical solutions for beach aquifers with depth-dependent K. Each of these contributions either relied on numerical computation or employed transform techniques tied to particular boundary configurations. The governing PDE was not written in a canonical form that would allow a systematic classification of its invariance properties.

On the African continent, Comte et al. [9] calibrated depth-dependent hydraulic conductivity profiles for the Nairobi Aquifer System, confirming that K declines with depth, and Ekwe et al. [2] reached a similar conclusion for the middle Imo River basin in Nigeria using geophysical sounding. Neither study cast the conductivity-depth relationship into a differential equation. A study of groundwater constraints in Kenyan urban areas [8] and the East African region [7, 10] noted the lack of analytical frameworks linking velocity to depth.

The objective of this paper is to derive the governing PDE for each of the two conductivity models from first principles, to express both equations in a single standard form that makes their coefficient structure transparent, and to compare the structural features that distinguish them.

II. MATHEMATICAL FORMULATION

2.1 Darcy's law and the continuity equation

Let h(x,z,t) denote the hydraulic head at horizontal position x, vertical depth z, and time t in a confined aquifer whose hydraulic conductivity K depends on z alone. The volumetric flux per unit area is given by Darcy's law,

$$\mathbf{q} = -\frac{K}{\mu}\nabla h$$

with μ the dynamic viscosity of water. The horizontal component of the Darcy velocity at depth z is therefore

$$v(z) = -\frac{K(z)}{\mu}\frac{\partial h}{\partial x}$$

where x is aligned with the dominant flow direction. Mass conservation for an incompressible fluid in a saturated medium with specific storage [3, 1] S_s requires

$$\frac{\partial}{\partial x}\left(K(z)\frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial z}\left(K(z)\frac{\partial h}{\partial z}\right) = S_s\frac{\partial h}{\partial t}$$

Here the problem is restricted to the vertical–longitudinal plane (x,z); lateral flow in the y-direction is omitted. This two-dimensional idealisation is standard for confined systems in which a single horizontal gradient dominates [3] .

2.2 Conductivity models

Two one-parameter families of depth-dependent conductivity are considered. The first is the exponential decay

$$K(z) = K_0 e^{-\alpha z}, \qquad \alpha > 0$$

in which K_0 is the reference conductivity at z=0 and α is the decay rate. This form models the monotonic pore-space closure produced by increasing overburden [6, 12] . The second is the power-law decay

$$K(z) = K_0 z^{-\beta}, \qquad \beta > 0$$

which represents a slower, scale-dependent reduction appropriate for formations with irregular layering [11] .

Six standing assumptions apply:

1. the aquifer is confined, saturated, and of uniform thickness L;
2. the fluid is incompressible with constant viscosity μ;
3. K varies only with z;
4. flow is two-dimensional in the x-z plane;
5. no internal sources or sinks are present;
6. K_0, α, β, and S_s are positive constants.

III. DERIVATION OF THE GOVERNING PDES

3.1 Exponential decay case

Inserting $K(z) = K_0 e^{-\alpha z}$ into the left-hand side of

([eq:continuity]) and noting that K does not depend on x,

$$K_0 e^{-\alpha z}\frac{\partial^2 h}{\partial x^2} + \frac{\partial}{\partial z}\left(K_0 e^{-\alpha z}\frac{\partial h}{\partial z}\right) = S_s\frac{\partial h}{\partial t}$$

The product in the second term is differentiated by Leibniz's rule:

$$\frac{\partial}{\partial z}\left(K_0 e^{-\alpha z}\frac{\partial h}{\partial z}\right) = K_0\frac{\partial}{\partial z}\left(e^{-\alpha z}\right)\frac{\partial h}{\partial z} + K_0 e^{-\alpha z}\frac{\partial^2 h}{\partial z^2}$$

Since $\frac{\partial}{\partial z}(e^{-\alpha z}) = -\alpha e^{-\alpha z}$ equation ([eq:product_exp]) becomes

$$\frac{\partial}{\partial z}\left(K_0 e^{-\alpha z}\frac{\partial h}{\partial z}\right) = -\alpha K_0 e^{-\alpha z}\frac{\partial h}{\partial z} + K_0 e^{-\alpha z}\frac{\partial^2 h}{\partial z^2}$$

Returning this to ([eq:sub_exp]) and extracting the common factor $K_0 e^{-\alpha z}$,

$$K_0 e^{-\alpha z}\left(\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial z^2} - \alpha\frac{\partial h}{\partial z}\right) = S_s\frac{\partial h}{\partial t}$$

The prefactor $K_0 e^{-\alpha z}$ is strictly positive for every $z \geq 0$, so both sides may be divided by it to obtain

$$\boxed{\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial z^2} - \alpha\frac{\partial h}{\partial z} = \frac{S_s}{K_0}e^{\alpha z}\frac{\partial h}{\partial t}}$$

The left-hand side of ([eq:standard_exp]) contains a constant first-order coefficient $-\alpha$ multiplying h_z,which functions as a depth-directed convection arising from the spatial gradient of K. On the right-hand side the multiplier $(S_s/K_0)\,e^{\alpha z}$ increases with depth, indicating that the effective diffusivity $K_0 e^{-\alpha z}/S_s$ diminishes and head disturbances propagate more slowly at greater depth.

3.2 Power-law decay case

With $K(z) = K_0 z^{-\beta}$ in ([eq:continuity]),

$$K_0 z^{-\beta}\frac{\partial^2 h}{\partial x^2} + \frac{\partial}{\partial z}\left(K_0 z^{-\beta}\frac{\partial h}{\partial z}\right) = S_s\frac{\partial h}{\partial t}$$

Applying Leibniz's rule to the second term,

$$\frac{\partial}{\partial z}\left(K_0 z^{-\beta}\frac{\partial h}{\partial z}\right) = K_0\frac{\partial}{\partial z}(z^{-\beta})\frac{\partial h}{\partial z} + K_0 z^{-\beta}\frac{\partial^2 h}{\partial z^2}$$

Using $\frac{\partial}{\partial z}(z^{-\beta}) = -\beta z^{-\beta-1}$,

$$\frac{\partial}{\partial z}\left(K_0 z^{-\beta}\frac{\partial h}{\partial z}\right) = -\beta K_0 z^{-\beta-1}\frac{\partial h}{\partial z} + K_0 z^{-\beta}\frac{\partial^2 h}{\partial z^2}$$

Substituting back and collecting the $K_0 z^{-\beta}$ factor,

$$K_0 z^{-\beta}\left(\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial z^2} - \frac{\beta}{z}\frac{\partial h}{\partial z}\right) = S_s\frac{\partial h}{\partial t}$$

Division by $K_0 z^{-\beta}$ is valid for $z > 0$ and gives

$$\boxed{\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial z^2} - \frac{\beta}{z}\frac{\partial h}{\partial z} = \frac{S_s}{K_0}z^{\beta}\frac{\partial h}{\partial t}}$$

Here the first-order coefficient $-\beta/z$ is no longer constant; it blows up as $z \to 0^+$, reflecting the fact that the power-law model itself diverges at the origin and is physically meaningful only for $z > 0$. The right-hand multiplier $(S_s/K_0)\,z^{\beta}$ grows as a power of depth, so the effective diffusivity decreases more gently than in the exponential model.

IV. STEADY-STATE SPECIALISATIONS

When all time derivatives vanish $(h_t = 0)$, the right-hand sides of both ([eq:standard_exp]) and ([eq:standard_pow]) drop out.

The exponential equation reduces to

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial z^2} - \alpha\frac{\partial h}{\partial z} = 0$$

All coefficients in ([eq:steady_exp]) are constants, so the equation belongs to the constant-coefficient elliptic class. The variable factor $e^{\alpha z}$ that multiplied

h_t is now absent, and one may anticipate a wider family of invariant transformations than in the transient problem.

The power-law equation reduces to

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial z^2} - \frac{\beta}{z}\frac{\partial h}{\partial z} = 0$$

The β/z coefficient persists, keeping the equation in the variable-coefficient class. Equation ([eq:steady_pow]) falls within the family of generalised axisymmetric potential equations [4].

V. UNIFIED STRUCTURE AND COMPARATIVE ANALYSIS

Equations ([eq:standard_exp]) and ([eq:standard_pow]) are both instances of a single two-parameter family:

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial z^2} - f(z)\frac{\partial h}{\partial z} = g(z)\frac{\partial h}{\partial t}$$

The pair (f,g) encodes the conductivity model completely (Table 1).

Coefficient functions and structural properties for the two conductivity models.

Property	Exponential $K = K_0 e^{-\alpha z}$	Power-law $K = K_0 z^{-\beta}$
$f(z)$	α (constant)	β/z (singular at $z = 0$)
$g(z)$	$(S_s/K_0)\, e^{\alpha z}$	$(S_s/K_0)\, z^{\beta}$
Growth of g	Exponential	Algebraic
Steady-state type	Constant-coefficient	Variable-coefficient
Domain	$z \geq 0$	$z > 0$

Three contrasts stand out.

(i) Nature of the first-order coefficient. A constant $f = \alpha$ leaves the exponential PDE invariant under vertical translations $z \mapsto z + \delta$; the equation has the same form at every depth. The singular coefficient $f = \beta/z$ in the power-law PDE breaks this translational symmetry, because shifting z alters the coefficient. When one later seeks Lie point symmetries, the exponential equation can admit a depth-shift generator while the power-law equation cannot [14] .

(ii) Rate of diffusivity decay. Writing the effective diffusivity as $D(z) = K(z)/S_s$, one has $D = (K_0/S_s)\, e^{-\alpha z}$ (exponential) and $D = (K_0/S_s)\, z^{-\beta}$ (power-law). For any fixed $\alpha > 0$ and $\beta > 0$ there exists a crossover depth beyond which the exponential diffusivity is smaller. This means that at large depths a pressure pulse in an exponential aquifer attenuates faster than in a power-law aquifer with comparable near-surface diffusivity.

(iii) Coefficient type under steady state. Setting $h_t = 0$ converts the exponential PDE into an equation with no z-dependent coefficients at all, whereas the power-law PDE retains β/z. Constant-coefficient elliptic equations generically possess larger symmetry groups than variable-coefficient ones [4] , so the exponential steady-state problem is expected to admit more independent invariant transformations.

Having obtained $h(x, z, t)$ from either equation, the horizontal velocity follows at once from ([eq:horizontal_v]):

$$v(z) = -\frac{K(z)}{\mu}\frac{\partial h}{\partial x}$$

The velocity-depth profile is therefore the product of a known function $(K(z)/\mu)$ and the horizontal head gradient (h_x), the latter being determined by the solution of the governing PDE.

VI. CONCLUSION

Starting from mass conservation and Darcy's law under two standard conductivity-depth models, this paper obtained the governing PDEs for hydraulic head in a confined aquifer with depth-dependent $K(z)$. Both equations were written in the single form $h_{xx} + h_{zz} - f(z)\,h_z = g(z)\,h_t$, with coefficient functions (f, g) determined entirely by the choice of K.

For exponential conductivity $K = K_0 e^{-\alpha z}$, the first-order coefficient $f = \alpha$ is constant and the time-derivative multiplier $g = (S_s/K_0)\,e^{\alpha z}$ grows without bound. For power-law conductivity $K = K_0 z^{-\beta}$, the first-order coefficient $f = \beta/z$ has a pole at the origin and $g = (S_s/K_0)\,z^{\beta}$ grows algebraically. In the steady-state limit the exponential equation becomes entirely constant-coefficient, while the power-law equation keeps its singular β/z term.

The three structural contrasts identified—constant versus singular first-order coefficient, exponential versus algebraic diffusivity growth, and constant-coefficient versus variable-coefficient steady-state form—determine which analytical techniques are applicable. In particular, the coefficient structure controls the Lie point symmetries admitted by each equation, the similarity reductions available, and the classes of special functions appearing in closed-form solutions. These consequences are explored in a companion paper.

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