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Numerical Two-Step Three Off-grid Hybrid Optimized Fourth derivative Methods for Solving fourth order Initial Value Problems Base on Volterra Integral Equation of the Second Kind

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Abstract- ***This article present A two-step three-off grid hybrid optimized fourth derivative methods for solving fourth order initial value problems base on volterra integral equation of the second kind, the method proposes a power series as the basis function for a selected three hybrid points which suitably optimizes one of the three off-grid points by equating the principal term of the local truncation error to zero and using the local truncation error to determine the approximate values of the unknown parameter by treating the other two as a free parameter, the basic properties of the method was scrutinize and the develop method is apply to work out some fourth order initial value problems of ordinary differential equations and from the numerical results obtained, it is observed that our new methods gives better approximation than the existing method compared with our result.***

Keywords: ***One Optimize point, Local Truncation error, Free- Parameter***

I. INTRODUCTION

Many physical and technical phenomena, such as structural mechanics, fluid dynamics, vibration analysis, control systems, and other applied sciences, are modeled by fourth-order ordinary differential equations. Considerable scientific interest has been devoted to solving fourth-order initial value problems (IVPs) of the form

$$\varphi^{(iv)} = \gamma\left(t, \varphi(t), \varphi'(t), \varphi''(t), \varphi'''(t)\right),\ \varphi(t_0) = \varphi_0, \varphi'(t_0) = \varphi'_0, \varphi''(t_0) = \varphi''_0, \varphi'''(t_0) = \varphi'''_0 \quad (1)$$

Fourth-order differential equations are typically converted into systems of first-order equations prior to the use of numerical techniques. However, as research by Kuboye et al. (2020) and Kayode and Adeyeye (2011) have shown, this reduction method frequently increases computational complexity and places additional demands on computer resources. A good number of researchers have suggested direct numerical methods for resolving fourth-order initial value problems (IVPs) without reducing them to equivalent first-order systems in order to get around these restrictions, including Awoyemi (2005), Kayode (2008), Adesanya et al. (2012), Adeyeye and Omar (2019), Akinnukawe and Odekunle (2023), Areo and Omole (2015), Jator (2008), and Mohammed (2010). Even though these approaches have shown a great deal of success, accuracy, computational efficiency, and numerical stability still need to be improved. Fourth-order IVPs can be solved using a variety of numerical approaches, from predictor-corrector techniques to hybrid and optimized strategies. As evidenced by the studies of Adesanya et al. (2013), Adebayo and Adebola (2016), and Adoghe and Omole (2019), hybrid approaches in particular have drawn increasing attention as useful substitutes for traditional direct methods because of their increased accuracy and flexibility. Optimized numerical techniques have been developed recently to increase solution accuracy and computational efficiency. For instance, While Bothayna and Sadeemr (2019), Joshua (2022), and Raymond et al. (2023, 2025) proposed a number of optimized hybrid and derivative-based schemes for the numerical solution of higher-order differential

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equations, Akinnukawe et al. (2024) introduced a novel fourth-order block algorithm for solving fourth-order initial value problems (IVPs).

This work creates a fourth-order two-step technique that optimizes one of the three hybrid points for the direct solution of fourth-order IVPs, driven by the need for increased accuracy and enhanced computational performance. The suggested method aims to increase numerical stability and solution accuracy by incorporating optimized hybrid points into the block scheme generation. To increase the method's overall efficiency and performance, one off-grid point is optimized while the other two are regarded as free parameters by reducing the principal term of the local truncation error and using it to estimate unknown parameters. The goal of the suggested strategy is to maintain a respectable area of absolute stability while achieving greater accuracy and computing efficiency than current techniques.

The structure of this article is as follows: In Section 2, the methodology and derivation of the proposed scheme are described; in Section 3, the fundamental properties of the method, including convergence and stability analyses, are examined; and in Section 4, numerical experiments on specific initial value problems are presented to evaluate the performance of the developed method, followed by discussions and conclusion.

II. DERIVATION OF THE METHOD

This section presents the development of an optimized numerical hybrid technique with power series for solving general fourth order second-kind Volterra integral equations using three off-grid points and a linear multistep algorithm. The collocation approach employs a power series as the approximate solution, and the resulting algorithm is formulated as follows

$$\mu(x) = \vartheta(x) + \sum_{j=0}^{s+r-1} \psi_i x^j \quad (2)$$

Let the approximate power series be of the form

$$\sum_{i=0}^{k} \tau_i \left(\mu_{n+i} - \vartheta_{n+i}\right) = h^4 \sum_{i=0}^{s+r-1} \varsigma_i(t) \psi_{n+i} \quad (3)$$

where $\vartheta_n = \vartheta(x_n), \mu_n = \mu(x_n)$ and $\tau_i, \varsigma_i \ (i = 0, 1, \cdots, m;\ j = i, \cdots, k)$ are constant coefficient of (3) to be determined which is considered the solution to the second kind volterra integral equation of the form

$$\sum_{i=0}^{m} \tau_i \left(\mu_{n+i} - \vartheta_{n+i}\right) = h^4 \sum_{i=0}^{k} \sum_{j=1}^{k} \beta_i^{(j)} k\left(x_{n+j}, x_{n+i}, \left(\mu_{n+j} - \vartheta_{n+j}\right)\right)\left(\mu_{n+i} - \vartheta_{n+i}\right)$$
(4)

it is important to note that (3) can be written as (2) subject to the following condition

$$\eta(x) = \mu(x) - \vartheta(x) \quad (5)$$

equation (2) is differentiated once we obtain

$$\mu'(t) - \vartheta'(t) = \sum_{i=0}^{s+r-1} i x^{i-1} \mu_j(t) \quad (6)$$

differentiating (2) for the second time we obtain

$$\mu''(t) - \vartheta''(t) = \sum_{i=0}^{s+r-1} i(i-1) x^{i-2} \psi_j(t) \quad (7)$$

Finally differentiating (2) for the third time we obtain

$$\mu'''(t) - \vartheta'''(t) = \sum_{i=0}^{s+r-1} i(i-1)(i-2) x^{j-3} \psi_j \quad (8)$$

Where $\psi_j \in \Re$ are real unknown values to be determined. We shall now consider three hybrid points a, b, c such that $0 < a < b < c < \cdots < N$ hold (Akinnukwe, 2024).

Evaluate (2), (5), (6) and (7) at and collocating (8) at $0, a, b, c, 1, 2$ points. The continuous approximation is then constructed by imposing three conditions which are

$$\left.\begin{array}{l}\left(\mu_{n+j}-\vartheta_{n+j}\right)=(\mu-\vartheta)\left(x_{n+j}\right),\ j=0,a,b,c,1,2\\ \left(\mu'_{n+j}-\vartheta'_{n+j}\right)=(\mu-\vartheta)'\left(x_{n+j}\right),\ j=0,a,b,c,1,2\\ \left(\mu''_{n+j}-\vartheta''_{n+j}\right)=(\mu-\vartheta)''\left(x_{n+j}\right),\ j=0,a,b,c,1,2\\ \left(\mu'''_{n+j}-\vartheta'''_{n+j}\right)=(\mu-\vartheta)'''\left(x_{n+j}\right),\ j=0,a,b,c,1,2\\ \left(\mu^{iv}_{n+j}-\vartheta^{iv}_{n+j}\right)=\varphi_{n+j},\ j=0,a,b,c,1,2\\ \xi=\frac{t-t_{n+4}}{h}\end{array}\right\} \tag{9}$$

Equation (2) result to system of nonlinear equation of the form

$$QG=V \tag{10}$$

$$\begin{bmatrix}1&0&0&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0&0&0\\0&0&0&1&0&0&0&0&0&0\\0&0&0&0&1&1&1&1&1&1\\0&0&0&0&0&\frac{a^1}{1!}&\frac{b^1}{1!}&\frac{c^1}{1!}&\frac{1^1}{1!}&\frac{2^1}{1!}\\0&0&0&0&0&\frac{a^2}{2!}&\frac{b^2}{2!}&\frac{c^2}{2!}&\frac{1^2}{2!}&\frac{2^2}{2!}\\0&0&0&0&0&\frac{a^3}{3!}&\frac{b^3}{3!}&\frac{c^3}{3!}&\frac{1^3}{3!}&\frac{2^3}{3!}\\0&0&0&0&0&\frac{a^4}{4!}&\frac{b^4}{4!}&\frac{c^4}{4!}&\frac{1^4}{4!}&\frac{2^4}{4!}\\0&0&0&0&0&\frac{a^5}{5!}&\frac{b^5}{5!}&\frac{c^5}{5!}&\frac{1^5}{5!}&\frac{2^5}{5!}\end{bmatrix}^{-1}\begin{bmatrix}\xi^0\\\frac{\xi^1}{1!}\\\frac{\xi^2}{2!}\\\frac{\xi^3}{3!}\\\frac{\xi^4}{4!}\\\frac{\xi^5}{5!}\\\frac{\xi^6}{6!}\\\frac{\xi^7}{7!}\\\frac{\xi^8}{8!}\\\frac{\xi^9}{9!}\end{bmatrix}=\begin{bmatrix}(\mu_n-\vartheta_n)\\(\mu'_n-\vartheta'_n)\\(\mu''_n-\vartheta''_n)\\(\mu'''_n-\vartheta'''_n)\\\varphi_n\\\varphi_{n+a}\\\varphi_{n+b}\\\varphi_{n+c}\\\varphi_{n+1}\\\varphi_{n+2}\end{bmatrix}$$

Solving (10) for ξ_i's by Gaussian elimination method to obtain the unknown values in term of the parameter a,b,c gives a continuous hybrid linear multistep method of the form

$$\eta(x)=\sum_{i=0}^{k}\tau_i\eta_{n+i}+h^4\left[\sum_{j=0}^{2}\psi_j\varphi_{n+j}+\psi_j\varphi_{n+a}+\psi_j\varphi_{n+b}+\psi_j\varphi_{n+c}\right] \tag{8}$$

Where $\tau_0=1$ $\tau_1=n$ $\tau_2=\frac{1}{2}n^2$ $\tau_3=\frac{1}{6}n^3$

$$\psi_0=-\frac{1}{30240}n^4\left(\frac{\begin{array}{l}-84n^2a+54n^3a-84n^2b-9n^4a+54n^3b-9n^4b-84n^2c+54n^3c-9n^4c+36n^3\\-24n^4+5n^5-126n^2ab+18n^3ab-126n^2ac+18n^3ac-126n^2bc+18n^3bc\\+252nab+252nac+252nbc-1260abc-42n^2abc+378nabc\end{array}}{abc}\right)$$

$$\psi_1=\frac{1}{15120}n^5\left(\frac{54n^2b-9n^3b+54n^2c-9n^3c-84nb-84nc+252bc+36n^2-27n^3+5n^4+18n^2bc-126nbc}{a(a-1)(a-2)(a-c)(a-b)}\right)$$

$$\psi_2=-\frac{1}{15120}n^5\left(\frac{54n^2a-9n^3a+54n^2c-9n^3c-84na-84nc+252ac+36n^2-27n^3+5n^4+18n^2ac-126nac}{b(b-1)(b-2)(b-c)(a-b)}\right)$$

$$\psi_3=\frac{1}{15120}n^5\left(\frac{54n^2b-9n^3b+54n^2a-9n^3a-84nb-84na+252ba+36n^2-27n^3+5n^4+18n^2ba-126nba}{c(c-1)(c-2)(a-c)(c-b)}\right)$$

$$\psi_4=\frac{1}{15120}n^5\left(\frac{36n^2a-9n^3a+36n^2c-9n^3c+36n^2b-9n^3b-18n^3+5n^4+18n^2ba+18n^2ac+18n^2bc-84nab-84nac-84nbc+252abc-42nabc}{(a-1)(b-1)(c-1)}\right)$$

$$\psi_5=-\frac{1}{15120}n^5\left(\frac{18n^2a-9n^3a+18n^2c-9n^3c+18n^2b-9n^3b-9n^3+5n^4+18n^2ba+18n^2ac+18n^2bc-42nab-42nac-42nbc+126abc-42nabc}{(a-2)(b-2)(c-2)}\right)$$

substituting $n=1$ in (8), a multistep formula to approximate the solution of (1.1) at the point t_{n+1} yields

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$$t_{n+1}=\left(\mu_n-\vartheta_n\right)+h\left(\mu'_n-\vartheta'_n\right)+\tfrac{1}{2}\left(\mu''_n-\vartheta''_n\right)h^2+\tfrac{1}{6}h^3\left(\mu'''_n-\vartheta'''_n\right)$$
$$+h^4\left[\begin{array}{l}\left(\frac{1}{30240}\frac{(39a+39b+39c-144ab-144ac-144bc+924abc-14)}{abc}\right)\varphi_n+\left(\frac{1}{15120}\frac{(-39b-39c+144bc+14)}{a(a-c)(a-1)(a-2)(a-b)}\right)\varphi_{n+a}\\ -\left(\frac{1}{15120}\frac{(-39a-39c+144ac+14)}{b(b-c)(b-1)(b-2)(a-b)}\right)\varphi_{n+b}+\left(\frac{1}{15120}\frac{(-39b-39a+144ba+14)}{c(a-c)(c-1)(c-2)(c-b)}\right)\varphi_{n+c}\\ +\left(\frac{1}{15120}\frac{(27a+27b+27c-66ab-66ac-66bc+210abc-13)}{(c-1)(a-1)(b-1)}\right)\varphi_{n+1}-\left(\frac{1}{30240}\frac{(9a+9b+9c-24ab-24ac-24bc+84abc-4)}{(c-2)(a-2)(b-2)}\right)\varphi_{n+2}\end{array}\right] \quad (9)$$

Moreover, upon substitution into the first derivative of equation (8), a linear multistep formula for approximating the solution of equation (1) at the specified point t'_{n+1} is obtained as follows:

$$ht'_{n+1}=h\left(\mu'_n-\vartheta'_n\right)+\left(\mu''_n-\vartheta''_n\right)h^2+\tfrac{1}{2}h^3\left(\mu'''_n-\vartheta'''_n\right)$$
$$+h^4\left[\begin{array}{l}\left(\frac{1}{3360}\frac{(22a+22b+22c-70ab-70ac-70bc+378abc-9)}{abc}\right)\varphi_n+\left(\frac{1}{1680}\frac{(-22b-22c+70bc+9)}{a(a-c)(a-1)(a-2)(a-b)}\right)\varphi_{n+a}\\ -\left(\frac{1}{1680}\frac{(-22a-22c+70ac+9)}{b(b-c)(b-1)(b-2)(a-b)}\right)\varphi_{n+b}+\left(\frac{1}{1680}\frac{(-22b-22a+70ba+9)}{c(a-c)(c-1)(c-2)(c-b)}\right)\varphi_{n+c}\\ +\left(\frac{1}{1680}\frac{(20a+20b+20c-42ab-42ac-42bc+112abc-11)}{(c-1)(a-1)(b-1)}\right)\varphi_{n+1}-\left(\frac{1}{3360}\frac{(6a+6b+6c-14ab-14ac-14bc+42abc-3)}{(c-2)(a-2)(b-2)}\right)\varphi_{n+2}\end{array}\right] \quad (10)$$

Furthermore, by substituting $m=1$ into the second derivative of equation (8), a multistep formula for approximating the solution of equation (1) at the point t''_{n+1} is obtained as follows:

$$h^2t''_{n+1}=\left(\mu''_n-\vartheta''_n\right)h^2+\tfrac{1}{2}h^3\left(\mu'''_n-\vartheta'''_n\right)$$
$$+h^4\left[\begin{array}{l}\left(\frac{1}{840}\frac{(21a+21b+21c-56ab-56ac-56bc+245abc-10)}{abc}\right)\varphi_n+\left(\frac{1}{420}\frac{(-21b-21c+56bc+10)}{a(a-c)(a-1)(a-2)(a-b)}\right)\varphi_{n+a}\\ -\left(\frac{1}{420}\frac{(-21a-21c+56ac+10)}{b(b-c)(b-1)(b-2)(a-b)}\right)\varphi_{n+b}+\left(\frac{1}{420}\frac{(-21b-21a+56ba+10)}{c(a-c)(c-1)(c-2)(c-b)}\right)\varphi_{n+c}\\ +\left(\frac{1}{420}\frac{(28a+28b+28c-49ab-49ac-49bc+105abc-18)}{(c-1)(a-1)(b-1)}\right)\varphi_{n+1}-\left(\frac{1}{840}\frac{(7a+7b+7c-14ab-14ac-14bc+35abc-4)}{(c-2)(a-2)(b-2)}\right)\varphi_{n+2}\end{array}\right] \quad (11)$$

Finally by substituting $m=1$ in the third derivative of (8), a multistep formula to approximate the solution of (1) at the point t'''_{n+1} yields

$$h^3 t''_{n+1} = h^3\left(\mu'''_n - \vartheta'''_n\right) + h^4 \left[\begin{array}{l} \left(\frac{1}{120}\frac{(7a+7b+7c-15ab-15ac-15bc+50abc-4)}{abc}\right)\varphi_n + \left(\frac{1}{60}\frac{(-7b-7c+15bc+4)}{a(a-c)(a-1)(a-2)(a-b)}\right)\varphi_{n+a} \\ -\left(\frac{1}{60}\frac{(-7a-7c+15ac+4)}{b(b-c)(b-1)(b-2)(a-b)}\right)\varphi_{n+b} + \left(\frac{1}{60}\frac{(-7b-7a+15ba+4)}{c(a-c)(c-1)(c-2)(c-b)}\right)\varphi_{n+c} \\ +\left(\frac{1}{60}\frac{(18a+18b+18c-25ab-25ac-25bc+40abc-14)}{(c-1)(a-1)(b-1)}\right)\varphi_{n+1} - \left(\frac{1}{120}\frac{(3a+3b+3c-5ab-5ac-5bc+10abc-2)}{(c-2)(a-2)(b-2)}\right)\varphi_{n+2} \end{array} \right] \qquad (12)$$

2.2 Derivation of the Optimize Volterra Integral Equation of the second kind

Expanding (12) using the Taylor series to obtain the corresponding Local truncation error given as

$$L\left[y(x);h\right] = -\frac{1}{302400}\left(28a+28b+28c-49ab-49ac-49bc+105abc-18\right) + o(h^{11})$$
(13)

By equating the principal term of the local truncation errors in equation (13) to zero, and retaining as a free parameter by assigning $a = \frac{1}{4}$ & $b = \frac{1}{2}$ in equation (13), the value $c = \frac{5}{7}$ is obtained. Substituting the resulting values of a, b and c into equations (9)–(12) yields the optimized discrete scheme for the second-kind Volterra integral equations as follows

$$\begin{aligned}
\left(\mu_{n+\frac{1}{4}} - \vartheta_{n+\frac{1}{4}}\right) &= \left(\mu_n - \vartheta_n\right) + \tfrac{1}{4} h\left(\mu'_n - \vartheta'_n\right) + \tfrac{1}{32} h^2\left(\mu''_n - \vartheta''_n\right) + \tfrac{1}{384} h^3\left(\mu'''_n - \vartheta'''_n\right) + \tfrac{264251}{2477260800} h^4\varphi_n + \tfrac{106493}{1056706560} h^4\varphi_{n+\frac{1}{4}} \\
&\quad - \tfrac{10883}{139345920} h^4\varphi_{n+\frac{1}{2}} + \tfrac{4986877}{124216934400} h^4\varphi_{n+\frac{5}{7}} - \tfrac{1441}{212336640} h^4\varphi_{n+1} + \tfrac{4873}{93640458240} h^4\varphi_{n+2} \\
\left(\mu_{n+\frac{1}{2}} - \vartheta_{n+\frac{1}{2}}\right) &= \left(\mu_n - \vartheta_n\right) + \tfrac{1}{2} h\left(\mu'_n - \vartheta'_n\right) + \tfrac{1}{8} h^2\left(\mu''_n - \vartheta''_n\right) + \tfrac{1}{48} h^3\left(\mu'''_n - \vartheta'''_n\right) + \tfrac{3287}{2764800} h^4\varphi_n + \tfrac{547}{257985} h^4\varphi_{n+\frac{1}{4}} \\
&\quad - \tfrac{2671}{2177280} h^4\varphi_{n+\frac{1}{2}} + \tfrac{607453}{970444800} h^4\varphi_{n+\frac{5}{7}} - \tfrac{1219}{11612160} h^4\varphi_{n+1} + \tfrac{583}{731566080} h^4\varphi_{n+2} \\
\left(\mu_{n+\frac{5}{7}} - \vartheta_{n+\frac{5}{7}}\right) &= \left(\mu_n - \vartheta_n\right) + \tfrac{5}{7} h\left(\mu'_n - \vartheta'_n\right) + \tfrac{25}{98} h^2\left(\mu''_n - \vartheta''_n\right) + \tfrac{125}{2058} h^3\left(\mu'''_n - \vartheta'''_n\right) + \tfrac{67003625}{17432758224} h^4\varphi_n + \tfrac{2648900000}{297446437197} h^4\varphi_{n+\frac{1}{4}} \\
&\quad - \tfrac{35684375}{9805926501} h^4\varphi_{n+\frac{1}{2}} + \tfrac{5318875}{2548479024} h^4\varphi_{n+\frac{5}{7}} - \tfrac{18490625}{52298274672} h^4\varphi_{n+1} + \tfrac{8909375}{3294791304336} h^4\varphi_{n+2} \\
\left(\mu_{n+1} - \vartheta_{n+1}\right) &= \left(\mu_n - \vartheta_n\right) + h\left(\mu'_n - \vartheta'_n\right) + \tfrac{1}{2} h^2\left(\mu''_n - \vartheta''_n\right) + \tfrac{1}{6} h^3\left(\mu'''_n - \vartheta'''_n\right) + \tfrac{853}{75600} h^4\varphi_n + \tfrac{8096}{257985} h^4\varphi_{n+\frac{1}{4}} \\
&\quad - \tfrac{59}{8505} h^4\varphi_{n+\frac{1}{2}} + \tfrac{26411}{3790800} h^4\varphi_{n+\frac{5}{7}} - \tfrac{47}{45360} h^4\varphi_{n+1} + \tfrac{23}{2857680} h^4\varphi_{n+2} \\
\left(\mu_{n+2} - \vartheta_{n+2}\right) &= \left(\mu_n - \vartheta_n\right) + 2h\left(\mu'_n - \vartheta'_n\right) + 2h^2\left(\mu''_n - \vartheta''_n\right) + \tfrac{4}{3} h^3\left(\mu'''_n - \vartheta'''_n\right) + \tfrac{536}{4725} h^4\varphi_n + \tfrac{65536}{257985} h^4\varphi_{n+\frac{1}{4}} \\
&\quad + \tfrac{2048}{8505} h^4\varphi_{n+\frac{1}{2}} - \tfrac{19208}{236825} h^4\varphi_{n+\frac{5}{7}} + \tfrac{56}{405} h^4\varphi_{n+1} + \tfrac{214}{178605} h^4\varphi_{n+2}
\end{aligned}$$

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$$\left(\mu'_{n+\frac{1}{4}}-\vartheta'_{n+\frac{1}{4}}\right)=\left(\mu'_n-\vartheta'_n\right)+\tfrac{1}{4}h\left(\mu''_n-\vartheta''_n\right)+\tfrac{1}{32}h^2\left(\mu'''_n-\vartheta'''_n\right)+\tfrac{14111}{9175040}h^3\varphi_n+\tfrac{3613}{196864}h^3\varphi_{n+\frac{1}{4}}$$
$$-\tfrac{20987}{15482880}h^3\varphi_{n+\frac{1}{2}}+\tfrac{953197}{1380188160}h^3\varphi_{n+\frac{5}{7}}-\tfrac{3197}{27525120}h^3\varphi_{n+1}+\tfrac{4607}{5202247680}h^3\varphi_{n+2}$$
$$\left(\mu'_{n+\frac{1}{2}}-\vartheta'_{n+\frac{1}{2}}\right)=\left(\mu'_n-\vartheta'_n\right)+\tfrac{1}{2}h\left(\mu''_n-\vartheta''_n\right)+\tfrac{1}{8}h^2\left(\mu'''_n-\vartheta'''_n\right)+\tfrac{107}{13440}h^3\varphi_n+\tfrac{2027}{114660}h^3\varphi_{n+\frac{1}{4}}$$
$$-\tfrac{103}{12096}h^3\varphi_{n+\frac{1}{2}}+\tfrac{2401}{539136}h^3\varphi_{n+\frac{5}{7}}-\tfrac{121}{161280}h^3\varphi_{n+1}+\tfrac{39}{5080320}h^3\varphi_{n+2}$$
$$\left(\mu'_{n+\frac{5}{7}}-\vartheta_{n+\frac{5}{7}}\right)=\left(\mu'_n-\vartheta'_n\right)+\tfrac{5}{7}h\left(\mu''_n-\vartheta''_n\right)+\tfrac{25}{98}h^2\left(\mu'''_n-\vartheta'''_n\right)+\tfrac{1205375}{69177612}h^3\varphi_n+\tfrac{228040000}{4721372019}h^3\varphi_{n+\frac{1}{4}}$$
$$-\tfrac{2020000}{155649627}h^3\varphi_{n+\frac{1}{2}}+\tfrac{388625}{40452048}h^3\varphi_{n+\frac{5}{7}}-\tfrac{1350625}{830131344}h^3\varphi_{n+1}+\tfrac{81875}{6537284334}h^3\varphi_{n+2}$$
$$\left(\mu'_{n+1}-\vartheta'_{n+1}\right)=\left(\mu'_n-\vartheta'_n\right)+h\left(\mu''_n-\vartheta''_n\right)+\tfrac{1}{2}h^2\left(\mu'''_n-\vartheta'''_n\right)+\tfrac{1}{28}h^3\varphi_n+\tfrac{1088}{9555}h^3\varphi_{n+\frac{1}{4}}$$
$$-\tfrac{8}{945}h^3\varphi_{n+\frac{1}{2}}+\tfrac{2401}{84240}h^3\varphi_{n+\frac{5}{7}}-\tfrac{1}{336}h^3\varphi_{n+1}+\tfrac{1}{39690}h^3\varphi_{n+2}$$
$$\left(\mu'_{n+2}-\vartheta'_{n+2}\right)=\left(\mu'_n-\vartheta'_n\right)+2h\left(\mu''_n-\vartheta''_n\right)+2h^2\left(\mu'''_n-\vartheta'''_n\right)+\tfrac{5}{21}h^3\varphi_n+\tfrac{2048}{28665}h^3\varphi_{n+\frac{1}{4}}$$
$$+\tfrac{1216}{945}h^3\varphi_{n+\frac{1}{2}}-\tfrac{4802}{5265}h^3\varphi_{n+\frac{5}{7}}+\tfrac{202}{315}h^3\varphi_{n+1}+\tfrac{31}{3969}h^3\varphi_{n+2}$$

$$\left(\mu''_{n+\frac{1}{4}}-\vartheta''_{n+\frac{1}{4}}\right)=\left(\mu''_n-\vartheta''_n\right)+\tfrac{1}{4}h\left(\mu'''_n-\vartheta'''_n\right)+\tfrac{633}{40963}h^2\varphi_n+\tfrac{75}{2912}h^2\varphi_{n+\frac{1}{4}}-\tfrac{593}{34560}h^2\varphi_{n+\frac{1}{2}}+\tfrac{74431}{8626176}h^2\varphi_{n+\frac{5}{7}}$$
$$-\tfrac{59}{40960}h^2\varphi_{n+1}+\tfrac{253}{23224320}h^2\varphi_{n+2}$$
$$\left(\mu''_{n+\frac{1}{2}}-\vartheta''_{n+\frac{1}{2}}\right)=\left(\mu''_n-\vartheta''_n\right)+\tfrac{1}{2}h\left(\mu'''_n-\vartheta'''_n\right)+\tfrac{137}{3840}h^2\varphi_n+\tfrac{48}{455}h^2\varphi_{n+\frac{1}{4}}-\tfrac{7}{216}h^2\varphi_{n+\frac{1}{2}}+\tfrac{26411}{1347840}h^2\varphi_{n+\frac{5}{7}}-\tfrac{13}{3840}h^2\varphi_{n+1}+\tfrac{19}{725760}h^2\varphi_{n+2}$$
$$\left(\mu''_{n+\frac{5}{7}}-\vartheta''_{n+\frac{5}{7}}\right)=\left(\mu''_n-\vartheta''_n\right)+\tfrac{5}{7}h\left(\mu'''_n-\vartheta'''_n\right)+\tfrac{1042025}{19765032}h^2\varphi_n+\tfrac{40400000}{224827239}h^2\varphi_{n+\frac{1}{4}}-\tfrac{102500}{22235661}h^2\varphi_{n+\frac{1}{2}}+\tfrac{11650}{361179}h^2\varphi_{n+\frac{5}{7}}$$
$$-\tfrac{49375}{9883125}h^2\varphi_{n+1}+\tfrac{143125}{3735591048}h^2\varphi_{n+2}$$
$$\left(\mu''_{n+1}-\vartheta''_{n+1}\right)=\left(\mu''_n-\vartheta''_n\right)+h\left(\mu'''_n-\vartheta'''_n\right)+\tfrac{3}{40}h^2\varphi_n+\tfrac{128}{455}h^2\varphi_{n+\frac{1}{4}}+\tfrac{4}{135}h^2\varphi_{n+\frac{1}{2}}+\tfrac{2401}{21060}h^2\varphi_{n+\frac{5}{7}}+\tfrac{1}{22680}h^2\varphi_{n+2}$$
$$\left(\mu''_{n+2}-\vartheta''_{n+2}\right)=\left(\mu''_n-\vartheta''_n\right)+2h\left(\mu'''_n-\vartheta'''_n\right)+\tfrac{8}{15}h^2\varphi_n-\tfrac{2048}{1365}h^2\varphi_{n+\frac{1}{4}}+\tfrac{704}{135}h^2\varphi_{n+\frac{1}{2}}-\tfrac{4802}{1053}h^2\varphi_{n+\frac{5}{7}}+\tfrac{34}{15}h^2\varphi_{n+1}+\tfrac{26}{567}h^2\varphi_{n+2}$$

$$\left(\mu'''_{n+\frac{1}{4}}-\vartheta'''_{n+\frac{1}{4}}\right)=\left(\mu'''_n-\vartheta'''_n\right)+\tfrac{429}{5120}h\varphi_n+\tfrac{2629}{10920}h\varphi_{n+\frac{1}{4}}-\tfrac{2189}{17280}h\varphi_{n+\frac{1}{2}}+\tfrac{16807}{269568}h\varphi_{n+\frac{5}{7}}-\tfrac{79}{7680}h\varphi_{n+1}+\tfrac{223}{2903040}h\varphi_{n+2}$$
$$\left(\mu'''_{n+\frac{1}{2}}-\vartheta'''_{n+\frac{1}{2}}\right)=\left(\mu'''_n-\vartheta'''_n\right)+\tfrac{151}{1920}h\varphi_n+\tfrac{1448}{4095}h\varphi_{n+\frac{1}{4}}+\tfrac{13}{270}h\varphi_{n+\frac{1}{2}}+\tfrac{16807}{673920}h\varphi_{n+\frac{5}{7}}-\tfrac{31}{5760}h\varphi_{n+1}+\tfrac{17}{362880}h\varphi_{n+2}$$
$$\left(\mu'''_{n+\frac{5}{7}}-\vartheta'''_{n+\frac{5}{7}}\right)=\left(\mu'''_n-\vartheta'''_n\right)+\tfrac{32435}{403368}h\varphi_n+\tfrac{4672000}{13764933}h\varphi_{n+\frac{1}{4}}+\tfrac{79000}{453789}h\varphi_{n+\frac{1}{2}}+\tfrac{7685}{58968}h\varphi_{n+\frac{5}{7}}-\tfrac{12125}{1210104}h\varphi_{n+1}+\tfrac{5125}{76236552}h\varphi_{n+2} \quad (14)$$
$$\left(\mu'''_{n+1}-\vartheta'''_{n+1}\right)=\left(\mu'''_n-\vartheta'''_n\right)+\tfrac{3}{40}h\varphi_n+\tfrac{512}{1365}h\varphi_{n+\frac{1}{4}}+\tfrac{8}{135}h\varphi_{n+\frac{1}{2}}+\tfrac{16807}{42120}h\varphi_{n+\frac{5}{7}}+\tfrac{11}{120}h\varphi_{n+1}-\tfrac{1}{22680}h\varphi_{n+2}$$
$$\left(\mu'''_{n+2}-\vartheta'''_{n+2}\right)=\left(\mu'''_n-\vartheta'''_n\right)+\tfrac{17}{15}h\varphi_n-\tfrac{22528}{4095}h\varphi_{n+\frac{1}{4}}+\tfrac{1856}{135}h\varphi_{n+\frac{1}{2}}-\tfrac{67228}{5265}h\varphi_{n+\frac{5}{7}}+\tfrac{232}{45}h\varphi_{n+1}+\tfrac{661}{2835}h\varphi_{n+2}$$

III. ANALYSIS OF BASIC PROPERTIES OF THE METHOD

Let the linear operator associated with the discrete block method (14) be defined

3.1 Order of the Block

$$L\left\{\left(\mu_{n+j}-\vartheta_{n+j}\right):h\right\}=A^{(0)}\sum_{i=0}^{k}\tau_i\left(\mu_{n+j}-\vartheta_{n+j}\right)-\sum_{j=0}^{3}h^i e_i\left(\mu_n^{\,i}-\vartheta_n^{\,i}\right)-h^4\left[\sum_{j=0}^{2}\psi_j\varphi_{n+j}+\psi_j\varphi_{n+a}+\psi_j\varphi_{n+b}+\psi_j\varphi_{n+c}\right] \quad (15)$$

Expanding (12) in Taylor series and comparing the coefficient of h gives

$$L\{y(x):h\}=C_0y(x)+C_1y'(x)+,\ldots+C_ph^py^p(x)+C_{p+1}h^{p+1}y^{p+1}(x)+C_{p+2}h^{p+2}y^{p+2}(x)+C_{p+3}h^{p+3}y^{p+3}(x)+\ldots$$

Definition: Linear operator L and associated block formula are said to be of order p, if $C_0=C_1=,\ldots=C_p=C_{p+1}=C_{p+2}=C_{p+3}=0.$ and $C_{p+4}\neq 0.$ C_{p+4} is called the error constant and implies that the truncation error is given by $t_{n+k}=C_{p+3}h^{p+3}y^{p+3}(x)+O\left(h^{p+4}\right)$.

For our method, expanding (14) in Taylor series Comparing the coefficient of h gives $C_0=C_1=C_2=C_3=\ldots=C_6=0$ and

$$C_{10}=\begin{bmatrix}-\frac{37739}{39953262182400},-\frac{559}{39016857600},-\frac{48026875}{984044336228352},-\frac{89}{609638400},-\frac{2}{297675},-\frac{106493}{6658877030400}\\ -\frac{2671}{26011238400},-\frac{5318875}{23429627053056},-\frac{47}{101606400},-\frac{107}{3175200},-\frac{3613}{18496880640},-\frac{103}{216760320},\\ -\frac{388625}{557848263168},-\frac{1}{11258960},-\frac{31}{211680},-\frac{5}{3670016},-\frac{1}{1105920},-\frac{5825}{4980788064},\frac{8}{14175},-\frac{13}{30240}\end{bmatrix}$$

Hence our method is of order six (6).

3.2 Consistency

The optimize hybrid block method (14) is said to be consistent if it has an order more than or equal to one. Therefore, our method is consistent.

3.3 Zero Stability of Our Method

Definition 2:

An optimize fourth derivative hybrid block method is said to be zero-stable, if the roots $z_i, i=a,b,\ c,\ 1,2$ of the first characteristic polynomial $\rho(z)=0$ that is $\rho(z)=\det\left[\sum_{j=0}^{k}A^{(i)}z^{k-i}\right]=0$

Satisfies $|z_i|\leq 1$ and for those roots with z_i =1, multiplicity must not exceed two

Hence, our method is zero-stable.

3.2 Consistency

Theorem 3.1[6]

The necessary and sufficient conditions for the equation (14) to be convergent are that they must be consistent and zero-stable.

Therefore, the two-step fourth derivative method with one optimize off-grid hybrid points is convergent since it is both consistent and zero-stable

.

3.3 Linear Stability

The concept of A-stability according to Hairer E and Wanner G is discussed by applying the test equation

$$y^{(k)}=\lambda^k y,\quad k=4 \qquad (18)$$

To yield

$$Y_m=\varpi(z)Y_{m-1},z=\lambda h \qquad (19)$$

Where $\varpi(z)$ is the amplification matrix given by $\varpi(z)$

$$\varpi(z)=-\left(\xi^1-\varsigma^{(1)}-z^4\eta^{(1)}\right)^{-1}\left(\xi^{(0)}-\varsigma^{(0)}-z^4\eta^{(0)}\right) \qquad (20)$$

The matrix $\varpi(z)$ has Eigen values $(0,0,\cdots,\xi_k)$ where ξ_k is called the stability function.

Thus, the stability function for two-step optimize fourth derivative method with three off-grid hybrid points is given by

$$\xi = -\frac{945z^5 - 11121z^4 + 26280z^3 - 196020z^2 + 101880z - 100800}{25z^5 - 445z^4 + 4200z^3 - 23700z^2 + 75000z - 100800}$$

3.3 Regions of Absolute Stability

The stability polynomial for equation (14)

$$-h^5\left(\tfrac{1}{4032}w^5 + \tfrac{3}{320}w^4\right) + h^4\left(\tfrac{443}{6720}w^5 - \tfrac{89}{20160}w^4\right) - h^3\left(\tfrac{1}{24}w^5 + \tfrac{23}{84}w^4\right)$$
$$+h^2\left(\tfrac{79}{336}w^4 - \tfrac{251}{336}w^5\right) - h\left(\tfrac{125}{168}w^5 + \tfrac{211}{168}w^4\right) + w^5 - w^4$$

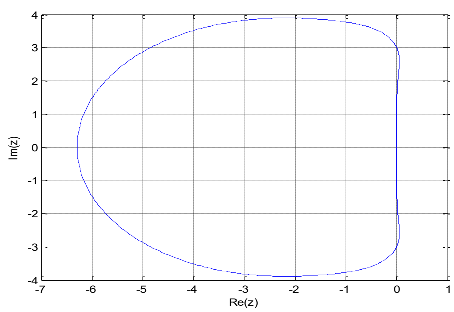

Figure 1. Showing the Region of Absolute Stability of (14)

IV. NUMERICAL RESULT

Problem 1 We solve the physical problem from ship dynamics. When a sinusoidal wave of frequency Ω passes long a ship or offshore structure, the resultant fluid actions vary with time t, therefore, consider the fourth-order problem as

$$\mu^{(iv)} + 3\mu'' + \mu\left(2 + \varepsilon\cos(\Omega t)\right) = 0, \quad t > 0,$$

Subjected to the following initial conditions $\mu(0) = 1$, $\mu'(0) = 0$, $\mu''(0) = \mu'''(0) = 0$, $h = \frac{1}{320}$

When $\varepsilon = 0$ for the existence of the theoretical solution: $\mu(t) = 2\cos(t) - \cos\left(t\sqrt{(2)}\right)$

Table 2. Showing the Result for Problem 1

x-values	Exact Solution	Error in our method	Error in Victor & Solomon (2021)
0.003125	0.99999999999205272181	0.000000	0.000000
0.00625	0.99999999987284392123	7.000000 e-20	1.110223e-16
0.009375	0.99999999935627549413	7.000000 e-20	0.000000
0.0125	0.99999999796552658061	6.000000 e-20	2.220446e-16
0.015625	0.99999999503306753346	6.000000 e-20	7.771561e-16
0.01875	0.99999998970067947566	3.000000 e-20	2.886580e-15
0.021875	0.99999998091947944409	4.000000 e-20	8.548717e-15
0.0250	0.99999996744995111885	9.000000 e-20	2.187139e-14
0.028125	0.99999994786198113954	1.000000 e-19	4.951595e-14
0.03125	0.99999992053490100510	1.000000 e-19	1.035838e-13

Problem 2 We consider the linear differential equation of fourth-order

$$\mu^{(iv)} + \mu'' = 0,$$

$\mu(0) = 0$, $\mu'(0) = \frac{-1.1}{72 - 50\pi}$, $\mu''(0) = \frac{1}{144 - 100\pi}$, $\mu'''(0) = \frac{1.2}{144 - 100\pi}$, $h = \frac{1}{100}$

with Exact Solution: $\mu(x) = \dfrac{1 - x - \cos(x) - 1.2\sin(x)}{144 - 100\pi}$,

Source: Blessing et al (2024)

Table 2. Showing the Result meant for Problem 2

x-values	Exact Solution	Error in our method	Error in Blessing *et al* (2024)	Error in Adeyefa & Olanegan (2024)
0.01	0.00012889983466749742	0.000000	3.80555e-17	2.988e-19
0.02	0.00025720540846480060	0.000000	2.69424e-17	2.391e-19
0.03	0.00038490976337573504	0.000000	6.55942e-17	8.070e-18

0.04	0.00051200600150551386	0.000000	3.20924e-17	1.913e-18
0.05	0.00063848728577052155	0.000000	3.14419e-17	3.736e-17
0.06	0.00076434684058201660	0.000000	3.22008e-17	6.456e-17
0.07	0.00088957795252368476	0.000000	3.45860e-17	1.025e-17
0.08	0.00101417397102297506	0.000000	2.51535e-17	1.530e-16
0.09	0.00113812830901615150	0.000000	6.87384e-17	2.179e-16
0.10	0.00126143444360699400	0.000000	1.56125e-16	2.989e-16

Problem 3 We consider the linear differential equation of fourth-order

$$\mu^{(iv)} - \mu = 0 ,$$

$$\mu(0) = 0 ,\ \mu'(0) = 0 ,\ \mu''(0) = -2,\ \mu'''(0) = 0,\ h = \frac{1}{320}$$

with Exact Solution:

$$\mu(x) = -\frac{1}{4}\exp(x) - \frac{1}{4}\exp(-x) + \frac{3}{2}\cos(x),$$

Source: Blessing et al (2024)

Table 1. Showing the Result meant for Problem 2

x-values	Exact Solution	Error in our method	Error in Blessing *et al* (2024)	Error in Blessing *et al* (2023)
0.003125	0.99999023437897364037	3.0000e-20	1.11022e-16	1.11022e-16
0.006250	0.99996093756357812224	9.00000e-20	2.22045e-16	1.11022e-16
0.009375	0.99991210969686319583	4.0000e-20	2.22045e-16	1.11022e-16
0.012500	0.99984375101724200780	8.0000e-20	1.11022e-16	1.11022e-16
0.015625	0.99975586185848644372	1.0000e-19	3.33067e-16	1.11022e-16
0.018750	0.99964844264972060957	5.0000e-19	1.11022e-16	1.11022e-16
0.021875	0.99952149391541244952	1.0000e-19	3.33067e-16	2.22045e-16
0.025000	0.99937501627536350206	2.0000e-19	4.44089e-16	1.11022e-16
0.028125	0.99920901044469679274	1.0000e-19	5.55112e-16	2.22045e-16
0.031250	0.99902347723384286572	3.0000e-19	4.44089e-16	0.00000+00

4.1 Conclusions

Several fourth-derivative numerical problems adopted from Akinnukawe (2024), Akinnukawe (2023), Victor and Solomon (2021), as well as Adeyefa and Olanegan (2024), was employed to evaluate the performance of the developed optimized two-step three hybrid fourth-derivative method base on Volterra Integral Equation of the second kind. The optimized hybrid methods were derived using Scientific Workplace 5.5 software, while implementation was carried out using Maple 18. In addition, MATLAB 2021a was utilized to generate the graphical representation, illustrating the L-stability region shown above. The results clearly demonstrate that the proposed optimized methods are capable of effectively solving fourth order equations and exhibit faster convergence compared to existing methods.

It is evident from the results presented in Tables 1, 2, and 3 that the proposed approach outperforms the existing method. Furthermore, if feasible, this approach may be extended to higher-order derivatives for solving general ordinary differential equations

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