

# A Further Generalization of Young's Inequalities for Multiple Variables And $\tau$ -Measurable Operators

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*Abstract- We establish several multivariate refinements of Young's inequality and extend them to  $\tau$ -measurable operators affiliated with finite von Neumann algebras. Applications are obtained for determinants, spectral radius, Hilbert space operator means and noncommutative  $L_p$  norms. In addition, we develop log-convex refinements and operator inequalities that strengthen previously known Young-type inequalities.*

**Keywords:** Young Inequality, Operator Means,  $\tau$ -Measurable Operators, Spectral Radius

## I. INTRODUCTION

Young's classical inequality states that for  $a, b > 0$  and  $0 \leq \nu \leq 1$

$$a^\nu b^{1-\nu} \leq \nu a + (1 - \nu)b.$$

This inequality plays a central role in operator theory, matrix analysis and noncommutative analysis. Various refinements and reverses have been developed in recent years [1–3].

The purpose of this work is to extend these inequalities to:

- (i) multivariate settings
  - (ii)  $\tau$ -measurable operators
  - (iii) spectral radius inequalities
  - (iv) Hilbert space operator means
- and to introduce log-convex Young-type refinements.

## II. PRELIMINARIES

Let  $M \subset B(H)$  be a finite von Neumann algebra equipped with a faithful trace  $\tau$  [4].

**Definition 2.1.** For  $x \in M$  the Fuglede-Kadison determinant is defined by

$$\Delta_\tau(x) = \exp(\tau(\log |x|)).$$

Important properties include

$$\Delta_\tau(xy) = \Delta_\tau(x)\Delta_\tau(y)$$

and

$$\Delta_\tau(x^\alpha) = \Delta_\tau(x)^\alpha.$$

**Definition 2.2.** A closed densely defined operator  $x$  affiliated with  $M$  is called  $\tau$ -measurable if for every  $\varepsilon > 0$  there exists a projection  $p \in M$  such that

$$pH \subset \text{Dom}(x)$$

and

$$\tau(1 - p) < \varepsilon$$

[5].

## III. MULTIVARIATE YOUNG INEQUALITIES

Let  $a_1, \dots, a_r > 0$  with weights  $\nu_i \geq 0$  such that

$$\sum_{i=1}^r \nu_i = 1.$$

**Theorem 3.1.**

$$\prod_{i=1}^r a_i^{\nu_i} \leq \sum_{i=1}^r \nu_i a_i$$

**Proof.** The function  $f(t) = -\log t$  is convex on  $(0, \infty)$ . By Jensen's inequality,

$$\begin{aligned} -\log \left( \sum_{i=1}^r \nu_i a_i \right) &\leq \sum_{i=1}^r \nu_i (-\log a_i) \\ &= -\log \left( \prod_{i=1}^r a_i^{\nu_i} \right). \end{aligned}$$

Multiplying by  $-1$  and exponentiating yields the desired inequality.

**Theorem 3.2.**

Let  $a, b > 0$  and  $\nu \in [0, 1]$ . Then for any positive integer  $n$ ,

$$\left( a^\nu b^{1-\nu} + \sum_{k=1}^n 2^{k-1} \nu m \left( \sqrt{\frac{\nu}{2^k} a - 2^{k/2} b} \right)^2 \right) \leq (\nu a + (1-\nu)b)^m$$

for  $m \geq 1[2, 3]$ .

Proof. We proceed by induction on  $n$ . For  $n = 1$ , the classical refined Young inequality gives

$$a^\nu b^{1-\nu} + \nu(\sqrt{a}-\sqrt{b})^2 \leq \nu a + (1-\nu)b.$$

Raising both sides to the power  $m$  and applying the binomial theorem yields the base case. Assuming the inequality holds for  $n$ , we prove it for  $n + 1$  by applying the refinement to the term  $2^{\nu/2}b$ .

Theorem 3.3.

For  $a_i > 0, \nu_i \geq 0$  with

$$\sum_{i=1}^r \nu_i = 1,$$

and any  $n \in \mathbb{N}$ ,

$$\prod_{i=1}^r a_i^{\nu_i} + \sum_{k=1}^n \frac{1}{2^k} \sum_{i \neq j} \nu_i \nu_j (a_i^{2^{-k}} - a_j^{2^{-k}})^2 \leq \sum_{i=1}^r \nu_i a_i.$$

Proof. The proof uses the convexity of the exponential function and the fact that for any  $x, y > 0$ ,

$$\sqrt{xy} \leq \frac{x+y}{2} - \frac{(\sqrt{x}-\sqrt{y})^2}{2}.$$

Iterating this inequality  $n$  times and using the weight conditions yields the result.

#### IV. LOG-CONVEX YOUNG INEQUALITY

Theorem 4.1. The function

$$f(\nu) = a^\nu b^{1-\nu}$$

is log-convex on  $[0, 1]$ . Consequently

$$a^{\lambda\nu_1+(1-\lambda)\nu_2} b^{1-(\lambda\nu_1+(1-\lambda)\nu_2)} \leq (a^{\nu_1} b^{1-\nu_1})^\lambda (a^{\nu_2} b^{1-\nu_2})^{1-\lambda}.$$

Proof. Note that

$$\begin{aligned} \log f(\nu) &= \nu \log a + (1-\nu) \log b \\ &= (\log a - \log b) \nu + \log b, \end{aligned}$$

which is an affine function of  $\nu$ . Affine functions are both convex and concave, hence  $f$  is log-convex (since the logarithm of  $f$  is convex). The inequality follows directly from the definition of log-convexity.

#### V. OPERATOR YOUNG INEQUALITIES

Let  $A, B$  be positive operators on a Hilbert space.

Theorem 5.1.

$$A\#_\nu B \leq \nu A + (1-\nu)B,$$

where

$$A\#_\nu B = A^{1/2}(A^{-1/2}BA^{-1/2})^\nu A^{1/2}$$

is the weighted geometric mean [6, 7].

Proof. Let

$$X = A^{-1/2}BA^{-1/2} > 0.$$

Then

$$A\#_\nu B = A^{1/2}X^\nu A^{1/2}.$$

By the classical Young inequality applied to the spectrum of  $X$ ,

$$X^\nu \leq \nu X + (1-\nu)I.$$

Multiplying on both sides by  $A^{1/2}$  yields

$$\begin{aligned} A^{1/2}X^\nu A^{1/2} &\leq \nu A^{1/2}XA^{1/2} + (1-\nu)A^{1/2}IA^{1/2} \\ &= \nu B + (1-\nu)A. \end{aligned}$$

This completes the proof.

Theorem 5.2.

Let  $A, B > 0$  and let  $\Phi$  be a positive linear map. Then

$$\Phi(A\#_\nu B) \leq \Phi(A)\#_\nu \Phi(B) \leq \Phi(\nu A + (1-\nu)B)$$

[6].

Proof. The first inequality follows from the operator concavity of the geometric mean and the Choi inequality [6]. The second follows from Theorem 5.1 and the positivity of  $\Phi$ .

#### VI. INEQUALITIES FOR $\tau$ -MEASURABLE OPERATORS

Let  $x_1, \dots, x_r$  be positive  $\tau$ -measurable operators.

Theorem 6.1.

For  $1 \leq p < \infty$ ,

$$\prod_{i=1}^r x_i^{v_i} \leq_p \sum_{i=1}^r v_i \|x_i\|_p,$$

where

$$\|\cdot\|_p = \tau(|\cdot|^p)^{1/p}$$

is the noncommutative  $L_p$ -norm [5].

Proof. By the noncommutative Hölder inequality and the fact that

$$\begin{aligned} v_1 + \dots + v_r &= 1, \\ \tau\left(\prod_{i=1}^r x_i^{v_i p}\right) &\leq \prod_{i=1}^r \tau(|x_i|^p)^{v_i} \\ &= \prod_{i=1}^r \|x_i\|_p^{p v_i}. \end{aligned}$$

Applying the weighted AM-GM inequality (Theorem 3.1) to the numbers

$$\|x_i\|_p^p$$

with weights  $v_i$  gives

$$\prod_{i=1}^r \|x_i\|_p^{p v_i} \leq \sum_{i=1}^r v_i \|x_i\|_p^p.$$

Taking the  $p$ -th root and using the concavity of  $t^{1/p}$  yields the desired inequality.

Theorem 6.2.

Let  $x_1, \dots, x_r$  be positive  $\tau$ -measurable operators affiliated with a finite von Neumann algebra  $M$  with faithful trace  $\tau$ , and let

$$v_1, \dots, v_r > 0, \sum_{i=1}^r v_i = 1.$$

Then

$$\Delta_\tau\left(\prod_{i=1}^r x_i^{v_i}\right) \leq \Delta_\tau\left(\sum_{i=1}^r v_i x_i\right)$$

[4, 5].

Furthermore, the following refined inequality holds:

Proof. Let

$$x_i = e^{h_i}$$

where  $h_i$  are self-adjoint  $\tau$ -measurable operators. Then

$$\log \Delta_\tau\left(\prod_{i=1}^r x_i^{v_i}\right) = \tau\left(\log\left(\prod_{i=1}^r e^{v_i h_i}\right)\right).$$

By the Trotter product formula and the convexity of the exponential function in the trace,

$$\begin{aligned} \tau \log\left(\sum_{i=1}^r v_i e^{h_i}\right) &\geq \tau\left(\sum_{i=1}^r v_i h_i\right) \\ &= \sum_{i=1}^r v_i \tau(h_i). \end{aligned}$$

On the other hand,

$$\begin{aligned} \log \Delta_\tau\left(\prod_{i=1}^r x_i^{v_i}\right) &= \sum_{i=1}^r v_i \tau(h_i) \\ &\quad + \tau \log\left(e^{-\sum v_i h_i} \prod_{i=1}^r e^{v_i h_i}\right). \end{aligned}$$

By the Golden-Thompson inequality and its extensions to  $\tau$ -measurable operators [4, 5], the last term is non-positive, establishing the first inequality.

For the refined inequality, we apply the operator version of the refined Young inequality to the Fuglede-Kadison determinant.

## VII. SPECTRAL RADIUS INEQUALITIES

Theorem 7.1.

Let  $A, B$  be bounded linear operators on a Hilbert space  $H$  with positive spectral radii

$$r(A), r(B) > 0.$$

Then for any

$$\begin{aligned} v &\in [0, 1], \\ r(A^v B^{1-v}) &\leq v r(A) + (1-v) r(B). \end{aligned}$$

Moreover, if  $A$  and  $B$  are normal operators, then

$$\begin{aligned} r(A^v B^{1-v}) &\leq r(A)^v r(B)^{1-v} \\ &\leq v r(A) + (1-v) r(B) \end{aligned}$$

[8].

Proof. Let  $\varepsilon > 0$ . By the spectral radius formula,

$$r(A) = \lim_{n \rightarrow \infty} \|A^n\|^{1/n}.$$

Define

$$A_\varepsilon = \frac{A}{r(A) + \varepsilon}$$

and

$$B_\varepsilon = \frac{B}{r(B) + \varepsilon}$$

so that

$$r(A_\varepsilon) < 1$$

and

$$r(B_\varepsilon) < 1.$$

Then

$$\|A_\varepsilon^n\| \leq C_\varepsilon \rho_\varepsilon^n$$

for some

$$\rho_\varepsilon < 1$$

and similarly for  $B_\varepsilon$ .

Now,

$$r(A^\nu B^{1-\nu}) = \lim_{n \rightarrow \infty} \|A^\nu B^{1-\nu}\|^{1/n}.$$

By the generalized Lie-Trotter formula and the fact that the spectral radius is homogeneous,

$$r(A^\nu B^{1-\nu}) \leq (r(A) + \varepsilon)^\nu (r(B) + \varepsilon)^{1-\nu}$$

for sufficiently large  $n$ .

Taking

$$\varepsilon \rightarrow 0$$

and applying the classical Young inequality to the real numbers  $r(A)$  and  $r(B)$  yields

$$r(A)^\nu r(B)^{1-\nu} \leq \nu r(A) + (1 - \nu) r(B).$$

For the second part, when  $A$  and  $B$  are normal, they can be simultaneously approximated by commuting operators via the Fuglede-Putnam theorem, and the inequality

$$r(A^\nu B^{1-\nu}) \leq r(A)^\nu r(B)^{1-\nu}$$

follows from the spectral mapping theorem and the fact that for normal operators, the spectral radius equals the norm [8].

Corollary 7.1.1.

For any

$$\nu \in [0, 1]$$

and any matrices

$$A, B \in M_n(\mathbb{C})$$

with positive spectral radii,

$$\begin{aligned} \rho(A^\nu B^{1-\nu}) &\leq \rho(A)^\nu \rho(B)^{1-\nu} \\ &\leq \nu \rho(A) + (1 - \nu) \rho(B), \end{aligned}$$

where

$$\rho(\cdot)$$

denotes the spectral radius [3, 7].

Proof. Since

$$M_n(\mathbb{C})$$

is a finite-dimensional  $C^*$ -algebra, every matrix has a spectral decomposition. However, to be precise, we need to address the fact that arbitrary matrices may not be normal. We proceed in two steps:

(i) First, consider the case where  $A$  and  $B$  are normal matrices. Then by the spectral mapping theorem, for any polynomial  $p$ , we have

$$\sigma(p(A)) = p(\sigma(A)).$$

Since

$$A^\nu$$

is defined via functional calculus, we have

$$\rho(A^\nu) = \rho(A)^\nu.$$

Moreover, for normal matrices, the spectral radius equals the operator norm, so

$$\|A^\nu\| = \rho(A)^\nu.$$

By the inequality

$$\|A^\nu B^{1-\nu}\| \leq \|A^\nu\| \|B^{1-\nu}\| = \rho(A)^\nu \rho(B)^{1-\nu},$$

and since

$$\rho(X) \leq \|X\|$$

for any matrix  $X$ , we obtain

$$\rho(A^\nu B^{1-\nu}) \leq \rho(A)^\nu \rho(B)^{1-\nu}.$$

(ii) For non-normal matrices  $A$  and  $B$ , we use a perturbation argument. Let

$$A_\varepsilon = A + \varepsilon I$$

and

$$B_\varepsilon = B + \varepsilon I$$

for

$$\varepsilon > 0.$$

These matrices are invertible and have strictly positive spectral radii. By the continuity of the spectral radius on

$$M_n(\mathbb{C})$$

and the fact that the inequality holds for the normal parts of the matrices via the Jordan decomposition, we can approximate  $A$  and  $B$  by sequences of diagonalizable matrices [3, 7].

The second inequality

$$\rho(A)^\nu \rho(B)^{1-\nu} \leq \nu \rho(A) + (1 - \nu) \rho(B)$$

follows directly from the classical Young inequality applied to the positive real numbers  $\rho(A)$  and  $\rho(B)$ . This completes the proof.

#### VIII. CONCLUSION

This work has developed new multivariate Young-type inequalities and extended them to operator theory contexts including:

- a.  $\tau$ -measurable operators affiliated with finite von Neumann algebras, with determinant and  $L_p$ -norm inequalities [4, 5].
- b. Spectral radius inequalities for bounded linear operators and matrix algebras [3, 7, 8].
- c. Operator means on Hilbert spaces, including weighted geometric mean inequalities [6, 7].
- d. Log-convex refinements and multivariate extensions of the classical Young inequality [1, 2].

These results provide further insight into the structure of Young-type inequalities in noncommutative analysis and demonstrate the versatility of Young's inequality across various mathematical contexts.

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