Lie Symmetry Classification of The Groundwater Flow Equation with Exponential and Power-Law Depth-Dependent Hydraulic Conductivity in Confined Aquifers

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Abstract- The groundwater flow equation in a confined aquifer whose hydraulic conductivity decay with depth takes the form $h_{xx}+h_{zz}-f(z)\,h_z=g(z)\,h_t$, where the coefficient pair (f,g) is fixed by the conductivity model. This paper determines the complete Lie point symmetry algebras for four instances of this equation: the transient and steady-state versions under both exponential conductivity $K(z)=K_0\,e^{-\alpha z}$ and power-law conductivity $K(z)=K_0\,z^{-\beta}$. For each equation the second prolongation of the infinitesimal generator is applied, the resulting overdetermined system of determining equations is derived and solved, and the admitted generators are listed. The transient exponential equation admits a four-dimensional solvable algebra spanned by horizontal translation, a coupled depth-shift, time translation, and head scaling. The transient power-law equation also admits a four-dimensional solvable algebra, but the depth-shift is replaced by a self-similar dilation. In the steady-state limit the exponential algebra retains dimension four, with a rotation generator replacing time translation ($L\cong e(2)\oplus R$), whereas the power-law algebra contracts to dimension two. Commutator tables are constructed for every case, and the algebras are classified. The dependence of the symmetry type on the functional form of K(z) is discussed.

Keywords: Lie Point Symmetry, Infinitesimal Generator, Determining Equations, Confined Aquifer, Depth-Dependent Conductivity, Commutator Table, Lie Algebra Classification

I. INTRODUCTION

A one-parameter Lie group of point transformations maps solutions of a differential equation to other solutions whenever the equation is left invariant under the group action. The generators of such groups—the infinitesimal symmetries—can be found algorithmically by prolonging the generator to the appropriate jet space and requiring invariance on the solution manifold. This classical procedure, codified in [5], has been applied widely in fluid mechanics and mathematical physics but only sparingly in hydrogeology.

The groundwater flow equation for a confined aquifer with depth-dependent conductivity K(z) was derived in a companion paper [2] and written in the standard form

$$h_{xx}+h_{zz}-f(z)\,h_z=g(z)\,h_t$$

with $(f,g)=(\alpha, \lambda e^{\alpha z})$ for exponential conductivity and $(f,g)=(\beta/z, \lambda z^{\beta})$ for power-law conductivity, where $\lambda=S_s/K_0$. The two models produce PDEs with qualitatively different coefficient structures: the first has a constant convective term, the second a singular one.

Zhu et al. [8] showed that the symmetry algebra of Darcy-based flow equations depends on the functional form of K. Ibragimov [5] developed the prolongation apparatus and optimal-system theory used throughout the present work. Mkhonta [6] applied Lie symmetry methods to contaminant transport in saturated soils and obtained optimal systems and group-invariant solutions, but for a solute equation rather than the head equation. Saied and Khalifa [7] reduced an advection-diffusion equation in fractured rock using one-parameter groups, again for radionuclide transport rather than the velocity-depth problem. Abreu et al. [1] classified groundwater-related PDEs by their admitted symmetry groups without computing the generators explicitly for depth-dependent-K head equations in confined aquifers.

No previous study has determined the complete Lie point symmetry algebra for the head equation ([eq:standard]) under either the exponential or power-law conductivity model. The purpose of this paper is to carry out that determination for all four PDEs (two conductivity models × two temporal regimes), to construct the commutator tables, and to classify each algebra.

II. GOVERNING EQUATIONS

The four PDEs treated in this paper are stated below for reference. All notation follows [2] ; subscripts denote partial differentiation and $\lambda=S_s/K_0>0$.

Transient exponential:

$$\Delta_1 \equiv h_{xx}+h_{zz}-\alpha h_z-\lambda e^{\alpha z} h_t=0, \qquad z\ge 0$$

Steady-state exponential:

$$\Delta_2 \equiv h_{xx}+h_{zz}-\alpha h_z=0$$

Transient power-law:

$$\Delta_3 \equiv h_{xx}+h_{zz}-\frac{\beta}{z} h_z-\lambda z^{\beta} h_t=0, \qquad z>0$$

Steady-state power-law:

$$\Delta_4 \equiv h_{xx}+h_{zz}-\frac{\beta}{z} h_z=0$$

Every equation is linear and second-order. For a linear PDE the infinitesimal takes the form $\varphi=\sigma(x,z,t)\,h+\rho(x,z,t)$, where ρ satisfies the PDE itself and generates an infinite-dimensional ideal (the superposition symmetry) [5]. The finite-dimensional part is determined by the spatial infinitesimals ξ^1, ξ^2, the temporal infinitesimal τ, and the coefficient σ.

III. SYMMETRY ANALYSIS OF THE EXPONENTIAL EQUATIONS

3.1 Transient case: determining equations for Δ_1

Write the generator as $X=\xi^1\,\partial_x+\xi^2\,\partial_z+\tau\partial_t+(\sigma h+\rho)\,\partial_h$, with ξ^1, ξ^2, τ,σ depending on (x,z,t). Applying X^ [2] to Δ_1 and eliminating h_xx on the solution manifold via $h_{xx}=-h_{zz}+\alpha h_z+\lambda e^{\alpha z} h_t$, then splitting by independent monomials in the derivatives of h, yields the overdetermined system:

From h_zz: the Cauchy–Riemann pair

$$\xi^1_x=\xi^2_z, \qquad \xi^1_z=-\xi^2_x$$

From h_xt and h_zt:

$$\tau_x=0, \qquad \tau_z=0$$

so that $\tau=\tau(t)$.
From h_t (after using ([eq:CR])–([eq:tau_xz])):

$$\tau_t-2\xi^1_x-\alpha\xi^2=0$$

From h_x:

$$2\sigma_x+\alpha\xi^1_z-\lambda e^{\alpha z}\xi^1_t=0$$

From h_z:

$$2\sigma_z+\alpha\sigma-3\alpha\xi^2_z-\lambda e^{\alpha z}\xi^2_t=0$$

From h^0:

$$\sigma_{xx}+\sigma_{zz}-\alpha\sigma_z-\lambda e^{\alpha z}\sigma_t=0$$

3.2 Solution of the determining equations

The Cauchy–Riemann conditions ([eq:CR]) imply

that ξ^1 and ξ^2 are harmonic in (x,z):

$$\xi^1_{zz}=0 \text{ and } \xi^2_{xx}+\xi^2_{zz}=0.$$

In ([eq:hx_coeff]) the factor $\lambda e^{\alpha z}$ forces $\xi_t^1=0$; otherwise σ_x would acquire a z-dependence incompatible with harmonicity. Hence ξ^1=ξ^1 (x,z) and σ_x=-α/2 ξ_z^1=α/2 ξ_x^2.

By the same argument in ([eq:hz_coeff]), ξ_t^2=0. Equation ([eq:ht_coeff]) then reads τ_t=2ξ_x^1+αξ^2. The left side depends only on t and the right side only on (x,z), so both equal a constant c_1:

$$\tau = c_1 t + c_2, \qquad 2\xi_x^1 + \alpha\xi^2 = c_1$$

Differentiating the constraint 2ξ_x^1+αξ^2=c_1 with respect to z and x, and using the Cauchy–Riemann and harmonicity conditions, yields ξ_zz^2+α/2 ξ_z^2=0 and ξ_xz^2+α/2 ξ_x^2=0. The only harmonic function satisfying both is ξ^2=a_2 (constant), giving ξ^1=a_3 (constant) through the constraint. Hence c_1=αa_2 and τ=αa_2 t+c_2.

The remaining equations reduce to σ=α/2 a_2+c_0 (constant), verified in ([eq:h0_coeff]). The general finite-dimensional infinitesimal is therefore

$$X = a_3\,\partial_x + a_2\,\partial_z + (\alpha a_2 t + c_2)\,\partial_t + (\alpha a_2/2 + c_0)h\,\partial_h$$

3.3 Generators and algebra for the transient exponential equation

Setting each free constant to one in turn produces a basis for the admitted Lie algebra L_1:

$$\begin{aligned} 2X_1 &= \partial_x && \text{(horizontal translation)} \\ X_2 &= \partial_z + \alpha t\,\partial_t + \alpha/2\;h\,\partial_h && \text{(coupled depth-shift)} \\ X_3 &= \partial_t && \text{(time translation)} \\ X_4 &= h\,\partial_h && \text{(head scaling)} \end{aligned}$$

The commutator [X_i,X_j]=X_i X_j-X_j X_i is computed for each pair. All brackets involving X_1 vanish because X_1 has constant coefficients that depend on none of the arguments of X_2, X_3, X_4. The non-trivial brackets are

$$= \alpha\,X_3, \qquad [X_2, X_4] = \alpha/2\;X_4$$

These are displayed in Table 1.

Commutator table for the transient exponential algebra L_1.

$[\cdot,\cdot]$	X_1	X_2	X_3	X_4
X_1	0	0	0	0
X_2	0	0	αX_3	$\frac{\alpha}{2}X_4$
X_3	0	$-\alpha X_3$	0	0
X_4	0	$-\frac{\alpha}{2}X_4$	0	0

The derived series is L_1^((1)) =span {X_3, X_4}, L_1^ ((2)) = {0}, so L_1 is solvable. The ideal {X_3, X_4} is abelian, and X_2 acts on it by the diagonal matrix diag(α, α/2).

3.4 Steady-state exponential equation Δ_2

Dropping t from all infinitesimals and repeating the prolongation procedure on ([eq:Ess]), the Cauchy–Riemann conditions persist and the constraint ([eq:ht_coeff]) is replaced by 2ξ_x^1+αξ^2=0. The pair (ξ^1, ξ^2) is now permitted to be any harmonic pair satisfying this single linear relation, which admits the additional solution ξ^1=-z, ξ^2=x (rotation in the (x,z)-plane). The resulting four generators are:

$$\begin{aligned} 2Y_1 &= \partial_x, & Y_2 &= \partial_z + \alpha/2\;h\,\partial_h, \\ Y_3 &= -z\,\partial_x + x\,\partial_z + \alpha x/2\;h\,\partial_h, & Y_4 &= h\,\partial_h \end{aligned}$$

The commutator table is given in Table 2.

Commutator table for the steady-state exponential algebra L_2.

$[\cdot,\cdot]$	Y_1	Y_2	Y_3	Y_4
Y_1	0	0	Y_2	0
Y_2	0	0	$-Y_1$	0
Y_3	$-Y_2$	Y_1	0	0
Y_4	0	0	0	0

The sub-algebra $\{Y_1, Y_2, Y_3\}$ satisfies $[Y_1, Y_3] = Y_2$, $[Y_2, Y_3] = -Y_1$—the defining relations of e(2), the

Lie algebra of planar Euclidean motions. Since Y_4 commutes with every generator, L_2≅e (2) ⊕R.

IV. SYMMETRY ANALYSIS OF THE POWER-LAW EQUATIONS

4.1 Transient case: determining equations for Δ_3

Proceeding as in Section 3.1, but now with the coefficient β/z in place of α, the splitting yields the same Cauchy–Riemann pair ([eq:CR]) and τ=τ(t). The h_t-coefficient equation, however, takes the form

$$z(\tau_t - 2\xi_x^1) - \beta\,\xi^2 = 0$$

where the factor z multiplying τ_t-2ξ_x^1 distinguishes this system from ([eq:ht_coeff]).
The h_x and h_z equations again force ξ_t^1=ξ_t^2=0 through the λz^β factor. With ξ_x^2=0 (from the requirement that both sides of ([eq:ht_pow]) depend on z alone), the Cauchy–Riemann conditions give ξ_z^1=0 and ξ_x^1=ξ_z^2=c_1 (constant), so ξ^2=c_1 z+d and ξ^1=c_1 x+a_3.

Inserting into ([eq:ht_pow]): z τ_t=c_1 (2+β)z+βd. Matching powers of z: τ_t=c_1 (2+β) and βd=0. Since β>0, d=0: the power-law equation does not admit depth-translation, because the singularity at z=0 breaks translational invariance. The remaining infinitesimals reduce to σ=c_0 (constant), giving

$$\hat{X} = (c_1 x + a_3)\,\partial_x + c_1 z\,\partial_z + (c_1(2+\beta)t + c_2)\,\partial_t + c_0\,h\,\partial_h$$

4.2 Generators and algebra for the transient power-law equation

A basis for the algebra L_3 is

$$\begin{aligned}
2\hat{X}_1 &= \partial_x && \text{(horizontal translation)}\\
\hat{X}_2 &= x\,\partial_x + z\,\partial_z + (2+\beta)\,t\,\partial_t && \text{(self-similar dilation)}\\
\hat{X}_3 &= \partial_t && \text{(time translation)}\\
\hat{X}_4 &= h\,\partial_h && \text{(head scaling)}
\end{aligned}$$

Under the finite transformation generated by Xˆ_2, the coordinates scale as (x^*, z^*, t^*) = (e^ϵ x, e^ϵ z, e^(2+β) ϵ t). This dilation symmetry mirrors the scale-free character of z^(-β); by contrast the exponential equation admits a depth-shift (X_2) but no dilation.

The non-zero commutators are

$$= \hat{X}_1, \qquad [\hat{X}_2, \hat{X}_3] = -(2+\beta)\,\hat{X}_3$$

Commutator table for the transient power-law algebra L_3.

$[\cdot,\cdot]$	$\hat{X}_1$	$\hat{X}_2$	$\hat{X}_3$	$\hat{X}_4$
$\hat{X}_1$	0	$\hat{X}_1$	0	0
$\hat{X}_2$	$-\hat{X}_1$	0	$-(2+\beta)\hat{X}_3$	0
$\hat{X}_3$	0	$(2+\beta)\hat{X}_3$	0	0
$\hat{X}_4$	0	0	0	0

The derived series L_3^ ((1)) = {Xˆ_1, Xˆ_3}, L_3^ ((2)) = {0} shows that L_3 is solvable. The generator Xˆ_4 again forms a one-dimensional centre. Within the three-dimensional sub-algebra {Xˆ_1, Xˆ_2, Xˆ_3}, the dilation Xˆ_2 acts on Xˆ_1 with eigenvalue +1 and on Xˆ_3 with eigenvalue -(2+β), so the sub-algebra is an extension of the one-dimensional affine group.

4.3 Steady-state power-law equation Δ_4

With t absent, the constraint ([eq:ht_pow]) becomes 2zξ_x^1+βξ^2=0. The pair (ξ^1,ξ^2)=(a_3,0) (translation in x) solves this, but the rotation pair (ξ^1,ξ^2)=(-z,x) fails because -βx≠0, and the dilation pair (ξ^1,ξ^2)=(c_1 x,c_1 z) requires c_1 (2+β)z=0, forcing c_1=0.

Only two generators survive:

$$\hat{Y}_1 = \partial_x, \qquad \hat{Y}_2 = h\,\partial_h$$

The algebra L_4=span {Yˆ_1, Yˆ_2} is two-dimensional and abelian: [Yˆ_1, Yˆ_2] =0.

V. COMPARATIVE DISCUSSION

Table [tab:summary] collects the results.
Three observations follow.

(i) Shift versus dilation. The exponential conductivity e^(-αz) is unchanged (up to a multiplicative factor) under additive shifts z↦z+δ, producing the depth-shift generator X_2. The power-law conductivity z^(-β) is unchanged under multiplicative rescaling z↦e^ϵ z, producing the dilation generator Xˆ_2. This one-to-one correspondence between the invariance property of K(z) and the type of admitted generator is consistent with the general principle identified in [8].

(ii) Asymmetric steady-state behaviour. Removing t from the transient exponential equation eliminates X_3=∂_t but opens a new degree of freedom (rotation), so the algebra dimension stays at four. Removing t from the transient power-law equation eliminates both Xˆ_2 (which couples x, z, and t) and Xˆ_3=∂_t without compensation, halving the dimension to two. The difference originates in the coefficient structure: the steady-state exponential equation (([eq:Ess]) has constant coefficients, while the steady-state power-law equation ([eq:Pss]) retains the singular coefficient [5] β/z.

(iii) Common generators. Horizontal translation (∂_x) and head scaling (h∂_h) appear in every algebra. The first reflects the absence of explicit x-dependence in all four PDEs; the second reflects linearity. These two generators are the minimal symmetry content of any linear, x-independent, second-order head equation.

CONCLUSION

The Lie point symmetry algebras of the confined-aquifer head equation have been determined for exponential and power-law depth-dependent hydraulic conductivity under both transient and steady-state conditions. Additive shift-invariance of e^(-αz) produces a depth-shift generator; multiplicative scale-invariance of z^(-β) produces a dilation generator.

The exponential steady-state equation preserves the four-dimensional algebra by gaining a rotation in place of the lost time-translation, while the power-law steady-state equation collapses to a two-dimensional abelian algebra. These results provide the algebraic foundation for the symmetry reductions and closed-form solutions developed in companion papers [3, 4].

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