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Parameters of the Binary Linear Codes and Combinatorial Designs from the Alternating Groups A5, A7, and A9 in Terms of the Group Invariants

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***Abstract-** When a finite simple group acts on the costs of a maximal subgroup, the combinatorial designs and binary linear codes obtained from the action carry parameters that are governed by the invariants of the group. This paper characterises the parameters of the codes and designs constructed from the alternating groups A_5, A_7, and A_9 in terms of those invariants, namely the index of the subgroup, the subdegrees of the action, the conjugacy-class lengths, the centraliser orders, and the fixed-point counts. The length of every code is shown to equal the index of the subgroup, and the block sizes and replication numbers of every design are expressed in closed form through the orbit and class data, so that all parameters except the dimension are determined by the group invariants without reference to the incidence matrix. A counting formula for the fixed points of a class representative on a closet space is established and used to give the block sizes of the fixed-point construction. For constructions that combine several conjugacy classes, the dimension and minimum distance are bounded through the sum of the constituent subspaces. The parameters are tested against the Singleton, Griesmer, and Assmus–Key bounds, and the tightness of the dimension bound is assessed across the catalogues. The dimension is identified as the one parameter not fixed by the invariants alone, equal to the rank of the incidence matrix over the binary field, and the way in which the invariants distinguish the two non-conjugate PSL (2,8):C_3 classes of A_9 is made explicit.*

***Keywords:** Alternating Group, Binary Linear Code, Combinatorial Design, Group Invariant, Subdegree, Fixed Point, Parameter Bound.*

I. INTRODUCTION

A binary linear code of length n, dimension k, and minimum distance d is written $[n, k, d]$, and its usefulness is governed by these three parameters together with its weight distribution and automorphism group [5]. When a code arises from the action of a finite group on a combinatorial design, the parameters of the code and of the design are not arbitrary: they are constrained by the structure of the group, and in favourable cases they can be read off from invariants of the group without computing the code at all. Understanding which parameters are so determined, and which require direct computation, is the central question addressed here.

The codes and designs studied in this paper are those obtained from the alternating groups A_5, A_7, and A_9 by the point-stabiliser, conjugacy-class, and fixed-point methods of Key and Moori and of Moori [7, 9]. In a companion treatment the constructions themselves were carried out for every maximal subgroup and every non-identity conjugacy class of the three groups. The present paper takes those constructions as given and asks how their parameters depend on the group invariants, namely the index of the maximal subgroup, the subdegrees of the coset action, the conjugacy-class lengths, the centraliser orders of the class representatives, and the numbers of points fixed by those representatives.

The relevant invariants are classical. The index $[G:M]$ fixes the number of points on which the group acts; the subdegrees, the orbit lengths of a point stabiliser, fix the block sizes of the point-stabiliser construction; the class length and centraliser order, linked by the orbit-stabiliser relation $|g^G|\,|C_G(g)| = |G|$, fix the replication numbers; and the fixed-point count of a class representative fixes the block size of the fixed-point construction. The standing problem is to turn these qualitative dependencies into closed-form expressions and to determine the range within which they constrain the code parameters.

Although the construction methods have been applied to many groups, the parameters they produce have not been characterised uniformly in terms of the group invariants across A_5, A_7, and A_9, and the analysis for A_9 had not been carried out at all. A uniform characterisation has two benefits. It explains the regularities that recur across the three groups, such as the appearance of the even-weight code in every odd-degree natural action, and it isolates the dimension as the single parameter that the invariants do not determine, thereby clarifying exactly where computer-algebra computation is unavoidable [4].

This paper establishes the closed-form parameter expressions, proves a counting formula for the fixed points that underlies the fixed-point construction, develops the bounds that govern multi-class constructions, and tests the resulting parameters against the standard coding-theoretic bounds [1, 5]. The characterisation is applied throughout to the catalogues of A_5, A_7, and A_9, and is used to explain how the invariants separate the two non-conjugate $\mathrm{PSL}(2,8){:}C_3$ classes of A_9 that are fused in the symmetric group.

The remainder of the paper is organised as follows. Section 2 fixes notation and records the group invariants used. Section 3 gives the closed-form parameter expressions, Section 4 the invariant analysis and the fixed-point counting formula, and Section 5 the multi-class dimension and distance bounds. Section 6 tests the parameters against the classical bounds, and Section 7 treats the parameter-level duality and concludes.

II. PRELIMINARIES AND GROUP INVARIANTS

Throughout, G denotes one of the alternating groups A_5, A_7, or A_9, and M a maximal subgroup of G. The group acts on the set $\Omega = G/M$ of right cosets by right multiplication, and the action is faithful and primitive because M is maximal. The cardinality of Ω is the index $n = [G:M] = |G|/|M|$, and a fixed cost is written α, with stabiliser $G_\alpha = M$.

A $1\text{-}(v,k,\lambda)$ design $\mathcal{D} = (\mathcal{P},\mathcal{B})$ has a point set $\mathcal{P}$ of size v and a collection $\mathcal{B}$ of blocks, each a k-subset of $\mathcal{P}$, such that every point lies in exactly λ blocks. The number of blocks is b, and counting incidences in two ways gives the identity $bk = v\lambda$. The incidence matrix A of $\mathcal{D}$ has rows indexed by blocks and columns by points, with $A_{Bp} = 1$ when $p \in B$ and 0 otherwise. The binary code of the design is the row space $C(\mathcal{D}) = \langle A \rangle_{\mathbb{F}_2}$, of length v and dimension $\dim C(\mathcal{D}) = \mathrm{rank}_{\mathbb{F}_2} A$.

The constructions whose parameters are studied here are recalled briefly. In the point-stabiliser construction the blocks are the images under G of a non-trivial orbit of G_α on Ω. In the conjugacy-class construction, for a non-identity element g, the base block is the orbit of α under $\langle g \rangle$ and the block set is its G-orbit. In the fixed-point construction the base block is the set $\mathrm{Fix}_\Omega(g)$ of cosets fixed by g, again carried round by G. In every case G acts on the design and so $G \leq \mathrm{Aut}\big(C(\mathcal{D})\big)$.

The parameters of these designs and codes are expressed through the following invariants of G and its action on Ω.

Definition 1 (Group invariants of the action). For the action of G on $\Omega = G/M$ the relevant invariants are:

1. the *index* $n = [G\colon M]$, equal to $|\Omega|$;
2. the *subdegrees* $n_0 = 1, n_1, \ldots, n_{r-1}$, the lengths of the orbits of the point stabiliser G_α on Ω, where r is the rank of the action and $\sum_i n_i = n$;
3. for each conjugacy class g^G, the *class length* $|g^G|$ and the *centraliser order* $|C_G(g)|$, related by $|g^G|\,|C_G(g)| = |G|$;
4. the *fixed-point count* $|\mathrm{Fix}_\Omega(g)|$, the number of cosets fixed by g, equal to the permutation-character value of the action at g.

These invariants are read from the ATLAS [2] and verified in GAP [4]. The orbit-stabiliser relation in part (iii) is used repeatedly to pass between class lengths and centraliser orders, and the subdegrees in part (ii) coincide with the block sizes of the point-stabiliser construction. The fixed-point count in part (iv) is the quantity computed in closed form in Section 4. The length of every code constructed from Ω equals the index n, so the first invariant alone determines one of the three code parameters; the remaining sections determine the block sizes and replication numbers, and bound the dimension and minimum distance.

III. CLOSED-FORM PARAMETER EXPRESSIONS

The first of the three code parameters is fixed by a single invariant. Since the code spanned by a design on Ω has length equal to the number of points, and that number is the index of the maximal subgroup, the length is determined before any computation.

Proposition 2. Every code constructed from the action of G on Ω=G/M by the point-stabiliser, conjugacy-class, or fixed-point method has length n=[G:M] =|G|/|M|.

Proof. In each method the point set of the design is the whole closet space Ω, so the number of points is v=|Ω|=[G:M]. The code is the row space of the incidence matrix, whose columns are indexed by the points, so its length equals v=[G:M] =|G|/|M|.

The block sizes and replication numbers are equally explicit. In the point-stabiliser construction the base block is a non-trivial orbit of G_α, so its size is a subdegree n_i, the number of blocks equals the number of points, and the design is the symmetric 1-(n,n_i,n_i) design.

In the conjugacy-class construction the base block is the ⟨g⟩-orbit of α, of size equal to the length of that orbit, and the number of blocks is the size of the G-orbit of the base block; the replication number then follows from the incidence identity. In the fixed-point construction the base block is Fix_Ω (g), of size equal to the fixed-point count, and the number of blocks is |G|/|G_Δ |, where G_Δ is the set-wise stabiliser of the base block Δ=Fix_Ω (g), which equals |g^G | when the centraliser is that stabiliser.

In all cases the replication number is recovered from λ=bk/v, the standard counting identity for a 1-design. The resulting closed forms are collected in Table 1.

Table 1: Closed-form parameters of the three constructions in terms of the group invariants.

Method	Length v	Block size k	Number of blocks b	Replication λ
Point-stabiliser	$[G\colon M]$	n_i	$[G\colon M]$	n_i
Conjugacy-class	$[G\colon M]$	$\lvert\alpha^{\langle g\rangle}\rvert$	$\lvert G\rvert/\lvert G_\Delta\rvert$	bk/v
Fixed-point	$[G\colon M]$	$\lvert\mathrm{Fix}_\Omega(g)\rvert$	$\lvert G\rvert/\lvert G_\Delta\rvert$	bk/v

Table 1 records, for each construction method, the length, block size, number of blocks, and replication number of the resulting 1-design as functions of the group invariants of Definition 1. The length is the index in every row; the block size is a subdegree, a cyclic-orbit length, or a fixed-point count according to the method; the number of blocks is the index for the point-stabiliser method and an orbit length |G|/|G_Δ | for the other two; and the replication number is read from the identity [eq:rep].

The table makes precise which invariants control which parameters. The length and, for the point-stabiliser method, the replication number depend only on the index and the subdegrees, both of which are fixed once M is chosen, so these parameters are constant across all conjugacy classes. The block sizes of the other two methods depend on the chosen element g through the cyclic-orbit length or the fixed-point count, so they vary from class to class, but each remains a closed function of the invariants and requires no incidence computation.

Consequently, every design parameter in the table is determined by the group data alone.

This separation matches the pattern observed in the wider literature on codes from finite simple groups.

The point-stabiliser parameters reproduce the symmetric 1-(n,n_i,n_i) designs catalogued by Key, Moori, and Rodrigues [8] for the triangular-graph actions, and the fixed-point block sizes are the permutation-character values used by Moori [9]. The closed forms therefore extend to A_5, A_7, and A_9 the principle, established for sporadic and classical groups, that the design parameters are group invariants while only the code dimension calls for direct computation [1].

IV. GROUP-THEORETIC INVARIANT ANALYSIS

The block sizes of the fixed-point construction are governed by the number of cosets that an element fixes, and this number is itself a group invariant: it is the value at g of the permutation character of the action on G/M. The following formula expresses it through the centraliser orders.

Proposition 3. Let g∈G act on Ω=G/M. The number of cosets fixed by g is |Fix_Ω (g)|=|C_G (g)|∑_h^▒1/|C_M (h)|, where the sum is over a set of representatives h of the M-conjugacy classes contained in g^G∩M, and is zero when no conjugate of g lies in M.

Proof. A coset Mx is fixed by g if and only if Mxg=Mx, that is xgx^(-1) ∈M, so Fix_Ω (g) is non-empty only when some conjugate of g lies in M. The number of elements x with xgx^(-1)∈M, counted modulo M on the left, is obtained by partitioning g^G∩M into M-classes; the class of h contributes |C_G (g)|/|C_M (h)| fixed costs, since |C_G (g)|=|C_G (h)| for h conjugate to g in G. Summing over the M-class representatives h gives [eq:fix].

The subdegrees obey a complementary identity. Because the orbits of the point stabiliser G_α partition Ω, their lengths sum to the index, and their number is the rank of the action, which equals the number of G_α-orbits and hence the inner product of the permutation character with itself.

Proposition 4. For the action of G on Ω=G/M with subdegrees n_0=1,n_1,…,n_(r-1), ∑_(i=0)^(r-1)▒n_i =[G:M], and the rank r equals the number of orbits of G_α on Ω, which is the permutation-character inner product ⟨π,π⟩.

Proof. The orbits of G_α on Ω partition the [G:M] cosets, so their lengths sum to [G:M], giving [eq:subdeg]. The number of these orbits is by definition the rank r, and by Burnside's orbit-counting lemma applied to G_α, equivalently by the standard character identity, this number equals ⟨π, π⟩, where π is the permutation character of G on Ω. □

The subdegrees of the principal actions of the three groups are recorded in Table 2, where they are also seen to satisfy the identity [eq:subdeg].

Table 1: Subdegrees of representative actions of A_5, A_7, and A_9.

Group	Subgroup M	Index $[G:M]$	Subdegrees	Rank r
A_5	A_4	5	1,4	2
A_5	S_3	10	1,3,6	3
A_7	A_6	7	1,6	2
A_9	A_8	9	1,8	2
A_9	$\mathrm{PSL}(2,8){:}C_3$	120	1,56,63	3

Table 2 lists, for one or two representative maximal subgroups of each group, the index of the subgroup, the subdegrees of the action on its costs, and the rank. The natural actions of A_5, A_7, and A_9 on 5, 7, and 9 points are rank-2 with subdegrees 1 and $n-1$; the rank-3 actions of A_5 on 10 points and of A_9 on 120 points have three subdegrees each.

In every row the subdegrees sum to the index, confirming the identity [eq:subdeg]: $1+4=5$, $1+3+6=10$, $1+6=7$, $1+8=9$, and $1+56+63=120$. The rank-2 rows correspond to the 2-transitive natural actions, in which the single non-trivial subdegree $n-1$ is the block size of the point-stabiliser design, while the rank-3 rows produce two non-trivial designs each, of block sizes equal to the two larger subdegrees. The subdegrees thus determine the block sizes of the point-stabiliser construction directly.

These subdegree patterns are consistent with the established theory of multiplicity-free and rank-3 permutation groups. The rank-3 action of A_5 on the ten duads is the Petersen-graph action whose subdegrees 1,3,6 appear in the triangular-graph codes of Key, Moori, and Rodrigues [8], and the rank-3 action of A_9 on 120 points, with subdegrees 1,56,63, is the largest treated here and underlies the duality analysed in Section 7. The fixed-point formula [eq:fix] and the subdegree identity [eq:subdeg] together supply the block sizes of all three constructions from the invariants alone [9], as in the analogous treatments of the Mathieu groups [3].

V. MULTI-CLASS CONSTRUCTION THEORY

A richer family of codes is obtained by combining the designs from several conjugacy classes on a single closet space. If $D_1\ldots, D_m$ are designs on the same point set Ω, their union $D=D_1\cup\cdots\cup D_m$ has block set the union of the block sets, and its code is the sum of the constituent codes.

Definition 5 (Combined construction). For designs D_1, D_m on Ω with codes $C_1\ldots, C_m$, the combined design has block multiset $\bigcup_i B(D_i)$ and code $C=C_1+\cdots+C_m=\langle A_1\ldots, A_m\rangle_{F_2}$, the row space of the stacked incidence matrices.

The replication number of the combined design adds, since each point inherits its incidences from every constituent.

Proposition 6. If the constituent designs $D_1\ldots$, D_m on Ω have replication numbers $\lambda_1\ldots$, λ_m, then the combined design has replication number $\lambda=\sum_{i=1}^{m} \lambda_i$.

Proof. Each point $p\in\Omega$ lies in exactly λ_i blocks of D_i. In the combined design the blocks through pare the disjoint union over i of those of each constituent, so p lies in $\sum_i \lambda_i$ blocks. As this count is independent of p, the combined design is a 1-design with replication $\sum_i \lambda_i$. □

The dimension and minimum distance of the combined code are controlled by the constituent codes through the elementary properties of a sum of subspaces.

Theorem 7. Let $C=C_1+\cdots+C_m$ be the combined code of constituent $[n,k_i,d_i]$ codes on Ω. Then $\max_i k_i \le \dim C \le \min(n, \sum_{i=1}^{m} k_i)$, and the minimum distance satisfies $d(C) \le \min_i d_i$. For two constituents the dimension is given exactly by $\dim C=k_1+k_2-\dim(C_1\cap C_2)$.

Proof. Each C_i is a subspace of C, so $\dim C\ge\dim C_i=k_i$ for every i, giving the lower bound. The sum of subspaces has dimension at most the sum of the dimensions and at most the ambient dimension n, giving the upper bound.

Since $C\supseteq C_i$, every non-zero codeword of C_i lies in C, so the least weight in C is at most the least weight in any C_i, which is [eq:distbound]. For $m=2$ the identity $\dim(C_1+C_2)=\dim C_1+\dim C_2-\dim(C_1\cap C_2)$ is the Grassmann formula for subspaces over F_2. □

The bounds are illustrated in Table 3 for the combination of the two designs from the rank-3 action of A_5 on ten points.

Table 1: Bounds for the combined code from the rank-3 action of A_5 on ten points.

Constituents	Parameters $[n,k_i,d_i]$	$dimC$ bound	$d(C)$ bound
C_1 (Petersen)	$[10,6,3]$	—	—
C_2 (complement)	$[10,4,4]$	—	—
$C = C_1 + C_2$	$[10,k,d]$	$6 \le k \le 10$	$d \le 3$

Table 3 takes as constituents the [10,6,3] Petersen code and the [10,4,4] complement code obtained from the two non-trivial subdegrees of the rank-3 action of A_5, and records the bounds that Theorem 7 places on their sum. The combined code C=C_1+C_2 has dimension at least 6 and at most 10, and minimum distance at most 3.

The bounds interpret the effect of combining classes. Because C contains the [10,6,3] constituent, its dimension cannot fall below 6 and its distance cannot exceed 3; combining designs therefore tends to raise the dimension towards the ambient value while lowering the distance towards the smallest constituent distance. The exact dimension within the interval is fixed by dim (C_1∩C_2), the one quantity in [eq:dimbound] that the invariants do not supply and that must be computed as a rank over F_2.

This behaviour is the parameter-level counterpart of the trade-off familiar throughout algebraic coding theory, in which enlarging a code raises its rate but lowers its distance [5]. The combined constructions thus interpolate between the constituent codes catalogued for A_5, A_7, and A_9, and the same bounds apply to the large multi-class codes from the index-120 action of A_9, where the constituent dimensions are larger and the combined dimension approaches the length [9].

VI. BOUNDS AND VERIFICATION

The parameters determined in the preceding sections can be tested against the classical bounds of coding theory, which relate the length, dimension, and minimum distance of any linear code. The first is the Singleton bound.

Proposition 8 (Singleton bound and the MDS case). Every [n,k,d] code satisfies d≤n-k+1. The even-weight codes [n, n-1,2] obtained from the natural actions of A_n for odd n attain this bound and are therefore maximum distance separable.

Proof. The Singleton bound d≤n-k+1 holds for any linear code, since deleting d-1 coordinates leaves a code of the same dimension. For the even-weight code [n, n-1,2] the bound reads 2≤n-(n-1) +1=2, which is met with equality, so the code is maximum distance separable. ☐

A second constraint is the Griesmer bound, n≥∑_(i=0) ^(k-1) ▒⌈ d/2^i⌉ for a binary [n,k,d] code, which a code meeting it with equality is said to attain. The even-weight code [n, n-1,2] gives ∑_(i=0) ^(n-2) ▒⌈ 2/2^i⌉=2+(n-2) =n, so it attains the Griesmer bound as well. The dimension itself is constrained by the Assmus–Key relation between a design and its code: the dimension equals the F_2-rank of the incidence matrix, and for a symmetric 1-design on n points this rank lies between the rank of the constituent circulant and the full value n. The verification of the catalogued parameters against these bounds is summarised in Table 4.

Table 2: Verification of representative codes against the Singleton bound.

Source action	$[n,k,d]$	Singleton $n-k+1$	MDS	Griesmer
A_5 natural	$[5,4,2]$	2	yes	attains
A_9 natural	$[9,8,2]$	2	yes	attains
A_5 rank-3	$[10,6,3]$	5	no	—
A_7 index	$[21,14,4]$	8	no	—

Source action	$[n,k,d]$	Singleton $n-k+1$	MDS	Griesmer
-21				
A_9 index -120	[120,8,56]	113	no	—

Table 4 records, for five representative codes from the three groups, the code parameters, the Singleton value n-k+1, whether the code is maximum distance separable, and whether it attains the Griesmer bound. The two even-weight codes from the natural actions meet both bounds, while the rank-3, index-21, and index-120 codes fall short of the Singleton value by margins that widen with the redundancy.

The table interprets the place of each code in the parameter landscape. The even-weight codes [5,4,2] and [9,8,2] are optimal in the strong sense of meeting the Singleton and Griesmer bounds simultaneously, which Proposition 8 explains as a property of the whole family [n, n-1,2]. The remaining codes trade distance for the symmetry and the large automorphism group supplied by the group action; the [120,8,56] code, in particular, is very far from the Singleton bound but has minimum distance 56 on only eight information bits, the high-redundancy regime in which the symmetry is the dominant design consideration rather than rate.

These findings are consistent with the literature on codes from finite simple groups, where optimality is typically reported for the small natural-action codes and the larger codes are valued instead for their structure and decodability [1, 5]. The optimal [21,14,4] code from A_7, although not maximum distance separable, is the best linear complementary dual code of its length and dimension, as reported by Kahkeshani [6], and the present verification confirms that the alternating-group catalogue contains both bound-attaining codes and high-symmetry codes of independent interest [9].

VII. PARAMETER DUALITY AND CONCLUSION

The invariants treated above also explain a duality between certain pairs of maximal subgroups. When a group has two maximal subgroups that are isomorphic but not conjugate, and that are exchanged by an automorphism, the invariants distinguish the two only on part of the class list, and this is what produces matching point-stabiliser catalogues alongside differing conjugacy-class catalogues.

Proposition 9 (Parameter duality). Let M and M' be non-conjugate maximal subgroups of G and let φ be an automorphism of G with φ(M)=M'. Then:

1. the subdegrees of G on G/M and on G/M' coincide, so the point-stabiliser parameters from M and from M' are identical;

2. the fixed-point counts satisfy |Fix_(G/M') (g)|=|Fix_(G/M) (φ^ (-1) (g)) | for every g, so they agree on every class fixed by φ and can differ only on classes moved by φ.

Proof. The map Mx↦M'φ(x) is a bijection from G/M to G/M' that intertwines the action of g on G/M with that of φ(g) on G/M', so the permutation characters satisfy π_M'=π_M∘φ^ (-1).

Since φ is an automorphism, the two actions are isomorphic as abstract permutation groups, so their subdegree multisets coincide, giving (i) and hence equal point-stabiliser parameters. The fixed-point count is the permutation-character value, so |Fix_(G/M') (g)|=π_M' (g)=π_M (φ^ (-1) (g)) =|Fix_(G/M) (φ^ (-1) (g)) |. When φ fixes the class of g these equals |Fix_(G/M) (g)|, and the counts can differ only when φ moves the class of g, which proves (ii). □

Corollary 10. The two non-conjugate A_9-classes M_4 and M_5, both isomorphic to PSL (2,8):C_3 and fused in S_9, have identical subdegrees 1,56,63 and hence identical point-stabiliser parameters. Their fixed-point counts differ only on the classes interchanged by the fusing odd permutation. The conjugacy-class code parameters were computed to differ on the classes 3C, 7A, 9A, and 12A.

Proof. An odd permutation t with tM_4 t^ (-1) =M_5 induces the automorphism φ=conj_t of A_9 with φ(M_4) =M_5. Proposition 9(i) gives the common subdegrees 1,56,63 and the identical point-stabiliser parameters, and part (ii) restricts the fixed-point divergence to the classes moved by t.

The divergence of the conjugacy-class code parameters on 3C, 7A, 9A, and 12A is the outcome of the direct computation over F_2, since the two coset spaces are inequivalent as G-sets and the cyclic-orbit structure on them need not correspond. □

The duality shows that the invariants determine the point-stabiliser parameters completely, even for non-conjugate subgroups, while the conjugacy-class parameters retain a genuine dependence on the choice of subgroup that only direct computation resolves.

This paper has characterised the parameters of the binary linear codes and combinatorial designs from A_5, A_7, and A_9 in terms of the group invariants. The length of every code equals the index of the maximal subgroup, and the block sizes and replication numbers of every design were expressed in closed form through the subdegrees, the cyclic-orbit lengths, the fixed-point counts, and the class lengths, so that all design parameters are fixed by the group data.

A counting formula for the fixed points was proved and used to supply the block sizes of the fixed-point construction, and the subdegree identity was used to supply those of the point-stabiliser construction. For combined constructions, the dimension and minimum distance were bounded through the sum of the constituent subspaces, the dimension lying between the largest constituent dimension and the ambient length, and the distance not exceeding the smallest constituent distance.

The parameters were tested against the Singleton and Griesmer bounds, which the even-weight codes from the natural actions were shown to attain, and against the Assmus–Key dimension relation.

Throughout, the dimension emerged as the single parameter not determined by the invariants alone, equal to the rank of the incidence matrix over the binary field and requiring direct computation. The parameter duality established for non-conjugate isomorphic maximal subgroups was specialised to the two PSL (2,8):C_3 classes of A_9, where the invariants give identical point-stabiliser parameters but distinguish the conjugacy-class catalogues.

Taken with the constructions on which it builds, the characterisation provides closed-form parameter formulae and reference bounds for the alternating-group codes and designs, and extends without modification to the next groups in the alternating series.

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