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Mathematical Analysis of a Stage-Structured Two-Patch Predator-Prey Model with Density-Dependent Prey Migration and Predator Maturation Delay

NEBERT KITUNI WAFULA[1], BONIFACE OTIENO KWACH[2], SAMUEL BONG'ANG'A APIMA[3]
[1,2]*Department of Mathematics, Kibabii University, Bungoma, Kenya*
[3]*Department of Mathematics and Statistics, Kaimosi Friends University, Kaimosi, Kenya*

Abstract- ***We propose and analyse a stage-structured predator-prey model in which prey are distributed across a refuge patch and a predation patch, with density-dependent fear-driven movement between them, while the predator population develops through two distinct life stages separated by a fixed maturation delay τ. Predation acts exclusively on exposed prey according to a Holling type II functional response. Non-negativity of solutions is established through quasi-positivity of the vector field, and exponential boundedness is demonstrated via Gronwall's inequality. The model admits three biologically meaningful equilibria: total extinction E_0, predator-free coexistence E_1, and interior coexistence E_2. The existence of E_1 follows from an application of the Intermediate Value Theorem to a cubic polynomial, while the coexistence equilibrium E_2 is expressed in explicit closed form for each component. The basic reproduction number R_0 is obtained by applying the next-generation matrix method with a delay-adjusted survival factor. Normalised sensitivity indices are derived analytically for every parameter; the conversion efficiency β and adult predator mortality d_2 each carry a sensitivity index of magnitude one, making them the dominant controls on R_0, whereas the delay-mortality product d_1 τ ranks as the next most influential quantity. Local asymptotic stability of E_1 and E_2 is determined through explicit Routh-Hurwitz conditions on the characteristic quasi-polynomial. Global asymptotic stability of E_1 when R_0<1 and of E_2 when R_0>1 is established using Lyapunov-Krasovskii functionals. The model undergoes a transcritical bifurcation at R_0=1, and a Hopf bifurcation of E_2 arises when τ surpasses the critical threshold τ_0^*, whose determination reduces to locating positive roots of a scalar cubic.***

Keywords: Predator-Prey, Stage Structure, Maturation Delay, Prey Refuge, Fear Effect

I. INTRODUCTION

Predator-prey dynamics have attracted mathematical scrutiny since the pioneering contributions of Lotka [1] and Volterra [2]. Two biologically grounded generalisations of this classical setting have continued to draw considerable interest: spatial heterogeneity through prey refuges, and temporal structure in predator development.

Prey refuges moderate predation pressure and can dampen oscillatory behaviour [3]. When predators are present in large numbers, prey tend to retreat more rapidly from the exposed zone to the refuge, a fear-driven response [4] that creates a nonlinear link between predator density and prey distribution. This coupling is absent in models where migration between patches proceeds at a fixed rate.

Predator models that distinguish between juveniles, who cannot hunt, and adults, who require a fixed period τ to reach maturity, produce qualitatively richer behaviour than unstructured models [5,6]. The maturation delay functions as an independent bifurcation parameter: once it crosses a critical threshold, the coexistence equilibrium loses stability and the system settles into sustained oscillations through a Hopf bifurcation [7,8]. In a recent contribution, Bhattacharjee, Roy, Acharjee, and Dutta [15] developed this programme further by considering a three-stage predator structure that incorporates juvenile hunting alongside maturation delay, deriving sharper criteria under which the delay drives periodic oscillations.

The Holling type II functional response [9] describes the saturating relationship between prey density and the consumption rate per predator, with its half-saturation term reflecting the constraint that handling time imposes on feeding [10]. Kalinkat, Rall, Uiterwaal, and Uszko [16] assembled broad empirical evidence showing that saturating responses of this form are widespread across natural predator–prey systems, lending strong biological support to the Holling type II term used here. Despite the ecological relevance of each of these components in isolation, no previous study has brought together a two-patch prey habitat with fear-driven migration, a stage-structured predator population with maturation delay, and a Holling type II predation term within a single unified framework.

The present paper pursues the following objectives: (i) to establish non-negativity and exponential boundedness of solutions; (ii) to derive the basic reproduction number R_0 using the next-generation matrix method; (iii) to obtain normalised sensitivity indices for every parameter; (iv) to prove local and global asymptotic stability for all biologically feasible equilibria; and (v) to characterise the transcritical and Hopf bifurcations the model can undergo.

The remainder of the paper is organised as follows. Section 2 presents the model formulation. Section 3 addresses well-posedness. Section 4 identifies and analyses the equilibria. Sections 5 and 6 derive R_0 and its sensitivity indices. Sections 7 and 8 treat local and global stability, respectively. Section 9 examines bifurcation behaviour. Section 10 offers concluding remarks.

II. MODEL FORMULATION

2.1 Model Assumptions

i. The prey population occupies two patches: a refuge (population $n_1(t)$, safe from predation) and a predation zone (population $n_2(t)$, exposed to adult predators).
ii. Both prey populations grow logistically with intrinsic rates r_i and carrying capacities k_i, $i = 1,2$.
iii. In the absence of predators, prey migrate from the refuge to the predation zone at the baseline rate a_0. When adult predators are present, prey migrate from the predation zone to the refuge at the density-dependent rate $a_1 p_2(t) + a_0$, where $a_1 > 0$ is the fear-induced migration coefficient.
iv. Juvenile predators $p_1(t)$ do not hunt; adult predators $p_2(t)$ hunt exclusively in the predation zone.
v. Predation follows the Holling type II functional response $\alpha n_2/(a_2 + n_2)$ with maximum rate α and half-saturation constant a_2.
vi. A fixed maturation delay $\tau > 0$ governs the juvenile-to-adult transition. A juvenile born at time $t - \tau$ survives to adulthood with probability $e^{-d_1 \tau}$.

2.2 Model Description and Equations

The model divides prey into a refuge patch $n_1(t)$ and a predation zone $n_2(t)$, and separates predators into juveniles $p_1(t)$ and adults $p_2(t)$. Each prey class grows logistically. Prey move from the refuge to the predation zone at the baseline rate a_0; the presence of adult predators induces additional density-dependent return movement at rate $a_1 p_2(t) + a_0$. Only adult predators hunt, consuming exposed prey through a Holling type II response $\alpha n_2 p_2 / a_2 + n_2$. Each prey item consumed converts to a juvenile predator at efficiency β; juveniles complete development and enter the adult stage after a fixed delay τ, subject to natural mortality at rates d_1 (juveniles) and d_2 (adults). These processes are captured by the following system:

$$
\begin{aligned}
\frac{dn_1}{dt} &= r_1 n_1(t)\left(1 - \frac{n_1(t)}{k_1}\right) - a_0 n_1(t) + (a_1 p_2(t) + a_0) n_2(t) \\
\frac{dn_2}{dt} &= r_2 n_2(t)\left(1 - \frac{n_2(t)}{k_2}\right) + a_0 n_1(t) - (a_1 p_2(t) + a_0) n_2(t) - \frac{\alpha n_2(t) p_2(t)}{a_2 + n_2(t)} \\
\frac{dp_1}{dt} &= \frac{\beta n_2(t) p_2(t)}{a_2 + n_2(t)} - \frac{\beta n_2(t) p_2(t-\tau)}{a_2 + n_2(t)} - d_1 p_1(t) \\
\frac{dp_2}{dt} &= \frac{\beta n_2(t) p_2(t-\tau)}{a_2 + n_2(t)} - d_2 p_2(t)
\end{aligned}
\quad (1)
$$

with constant history $(\theta) = (n_1^0, n_2^0, p_1^0, p_2^0) \geq \mathbf{0}$ for $\theta \in [-\tau, 0]$.

All parameters are strictly positive and $0 < \beta \leq \alpha$. The quantities r_1 and r_2 are the intrinsic growth rates of the refuge and predation-zone prey populations, with k_1 and k_2 their respective carrying capacities. The parameter a_0 is the baseline inter-patch movement rate, while a_1 is the fear-induced coefficient governing the density-dependent flux of prey returning from the predation zone to the refuge. The Holling type II response is determined by the maximum predation rate α and the half-saturation constant a_2. Each consumed prey item contributes to juvenile predator recruitment at conversion efficiency β, and d_1, d_2 denote the mortality rates of juvenile and adult predators, respectively. The parameter τ represents the fixed maturation delay, measured in months, that separates the juvenile and adult predator stages.

III. POSITIVITY AND EXPONENTIAL BOUNDEDNESS

Theorem 1 (Positivity). For any non-negative history $\phi \in C([-\tau, 0], \mathbb{R}_+^4)$, the unique solution of System ((1)) satisfies $n_1(t), n_2(t), p_1(t), p_2(t) \geq 0$, for all $t \geq 0$; that is, $\mathbb{R}_+^4$ is positively invariant.

Proof. The prey equations are handled by quasi-positivity. On $\{n_1 = 0\}$,

$$\dot{n}_1|_{n_1=0} = (a_1 p_2 + a_0)\, n_2 \geq 0, \qquad (2)$$

Since $a_0, a_1 > 0$ and $p_2, n_2 \geq 0$. On $\{n_2 = 0\}$,

$$\dot{n}_2|_{n_2=0} = a_0\, n_1 \geq 0, \qquad (3)$$

since $a_0 > 0$ and $n_1 \geq 0$. Hence the vector field points into $\mathbb{R}_+^4$ on both prey faces. For the predators, both equations are linear with non-negative forcing. Applying the variation of constants formula to the adult equation gives

$$p_2(t) = \phi_4(0)\, e^{-d_2 t} + \int_0^t e^{-d_2(t-s)} \frac{\beta n_2(s)\, p_2(s-\tau)}{a_2 + n_2(s)}\, ds,$$

(4)

which is non-negative since $\phi_4(0) \geq 0$ and $p_2(s-\tau) \geq 0$ for $s \in [0, \tau]$ by the initial history; induction on intervals of length τ extends this to all $t \geq 0$. For p_1, the variation of constants formula gives

$$p_1(t) = \phi_3(0)\, e^{-d_1 t} + \int_0^t e^{-d_1(t-s)} \frac{\beta n_2(s)}{a_2 + n_2(s)} [p_2(s) - p_2(s-\tau)]\, ds,$$

(5)

and $p_1(t) \geq 0$ follows from $\phi_3(0) \geq 0$, $p_2(s) \geq 0$, and the exponential bound on p_2 [12].

Definition 1 (Gronwall's Inequality [12]). Let z, φ, ψ be continuous on $[0, c]$ with φ non-decreasing and $\psi \geq 0$. If $z(t) \leq \varphi(t) + \int_0^t \psi(s) z(s)\, ds$, then $(t) \leq \varphi(t) \exp\left(\int_0^t \psi(s)\, ds\right)$. In particular, if $\varphi(t) = A > 0$ and $\psi(t) = \mu > 0$, then $z(t) \leq A e^{\mu t}$.

Theorem 2 (Exponential Boundedness). Set $C_0 = n_1(0) + n_2(0) + p_1(0) + p_2(0)$ and $\mu = \max\{r_1, r_2\}$. Every solution of System (1) satisfies

$0 \leq n_i(t),\ p_j(t) \leq C_0\, e^{\mu t}$ for all $t \geq 0$..

Proof. Let $V(t) = n_1 + n_2 + p_1 + p_2$. Adding the first and second equations of System (1), the migration and fear-driven flux terms cancel, yielding

$$\dot{n}_1 + \dot{n}_2 = r_1 n_1 (1 - n_1/k_1) + r_2 n_2 (1 - n_2/k_2) - \frac{\alpha n_2 p_2}{a_2 + n_2}.$$

Adding the third and fourth equations of System (1), the maturation-delay terms cancel, yielding

$$\dot{p}_1 + \dot{p}_2 = \frac{\beta n_2 p_2}{a_2 + n_2} - d_1 p_1 - d_2 p_2.$$

Combining and using $r_i(1 - n_i/k_i) \leq r_i$, $0 < \beta \leq \alpha$ (so $(\beta - \alpha)n_2 p_2/(a_2 + n_2) \leq 0$), and $d_1, d_2 > 0$:

$$\dot{V} \leq r_1 n_1 + r_2 n_2 \leq \mu(n_1 + n_2) \leq \mu V(t).$$

Integrating: $V(t) \leq V(0) + \mu \int_0^t V(s)\, ds$. Gronwall's inequality with $\varphi = C_0$ and $\psi = \mu$ gives $V(t) \leq C_0 e^{\mu t}$. Since $0 \leq n_i, p_j \leq V$, the component-wise bounds follow.

IV. EQUILIBRIUM ANALYSIS

Setting all time derivatives to zero and noting that $p_2(t - \tau) = p_2^*$ at equilibrium, the third equation in (1) reduces to $-d_1 p_1^* = 0$, giving $p_1^* = 0$ at every equilibrium. This is a structural feature of the stage-structured DDE formulation: the juvenile production and maturation fluxes cancel exactly at a constant steady state [5,11].

4.1 Total Extinction Equilibrium

$E_0 = (0,0,0,0)$ exists for all parameter values and is always unstable, since the positive intrinsic growth rates $r_i > 0$ produce a positive eigenvalue $= \max\{r_1, r_2\} - a_0 > 0$ whenever $\max(r_1, r_2) > a_0$.

4.2 Predator-Free Equilibrium

Setting $p_1^* = p_2^* = 0$ reduces the system to

$$\begin{aligned} r_1 n_1^*(1 - n_1^*/k_1) - a_0 n_1^* + a_0 n_2^* &= 0, \\ r_2 n_2^*(1 - n_2^*/k_2) + a_0 n_1^* - a_0 n_2^* &= 0. \end{aligned}$$

(6) - (7)

Adding ((6))–((7)) eliminates migration terms:

$$r_1 n_1^*(1 - n_1^*/k_1) + r_2 n_2^*(1 - n_2^*/k_2) = 0.$$

(8)

Rearranging ((6)) gives $n_2^* = n_1^*(\sigma + \varrho n_1^*)$ where $\sigma = 1 - r_1/a_0$ and $\varrho = r_1/(a_0 k_1)$. Substituting into ((8)) and dividing by $n_1^* > 0$ yields the cubic.

$$G(n_1^*) = -\frac{r_2 \varrho^2}{k_2} n_1^{*3} - \frac{2 r_2 \sigma \varrho}{k_2} n_1^{*2} + \left(r_2 \varrho - \frac{r_1}{k_1} - \frac{r_2 \sigma^2}{k_2}\right) n_1^* + (r_1 + r_2 \sigma) = 0.$$

(9)

Proposition 3 (Existence of E_1). If $a_0(r_1 + r_2) > r_1 r_2$, then G has at least one positive real root and the predator-free equilibrium $E_1 = (n_1^*, n_2^*, 0, 0)$ is biologically admissible.

Proof. The leading coefficient of G is $-r_2 \varrho^2/k_2 < 0$, so $G \to -\infty$ as $n_1^* \to +\infty$. At $n_1^* = 0$: $G(0) = r_1 + r_2 \sigma = r_1 + r_2 - r_1 r_2/a_0$. The condition $a_0(r_1 + r_2) > r_1 r_2$ is precisely $G(0) > 0$. By the Intermediate Value Theorem, G has at least one root in $(0, +\infty)$.. Once n_1^* is found, n_2^* is recovered from $n_2^* = n_1^*(\sigma + \varrho n_1^*)$.

4.3 Coexistence Equilibrium

For $p_2^* > 0$, the fourth equation of the equilibrium system gives

$$\frac{\beta n_2^*}{a_2 + n_2^*} = d_2 \quad \Rightarrow \quad n_2^* = \frac{a_2 d_2}{\beta - d_2}, \tag{10}$$

requiring the predator viability condition $\beta > d_2$. Biological admissibility further requires $n_2^* < k_2$, i.e. $k_2(\beta - d_2) > a_2 d_2$.

Set $N_2 = a_2 d_2/(\beta - d_2)$. With the identities $a_2 + N_2 = \beta a_2/(\beta - d_2)$ and $\alpha N_2 p_2^*/(a_2 + N_2) = (\alpha d_2/\beta) p_2^*$, the second equilibrium equation becomes

$$r_2 N_2(1 - N_2/k_2) + a_0 n_1^* - a_0 N_2 = (a_1 N_2 + \alpha d_2/\beta) p_2^*.$$

(11)

Define $\Gamma = a_1 N_2 + \alpha d_2/\beta > 0$ and $\Delta = N_2\left[r_2(1 - N_2/k_2) - a_0\right]$. Then

$$p_2^* = \frac{a_0 n_1^* + \Delta}{\Gamma}. \tag{12}$$

Substituting ((12)) into the first equilibrium equation and collecting powers of n_1^* yields the quadratic $\mathcal{A}n_1^{*2} - \mathcal{B}n_1^* - \mathcal{C} = 0,$ where

$$\mathcal{A} := \frac{r_1}{k_1}, \quad \mathcal{B} := r_1 - a_0 + \frac{a_0\, a_1\, N_2}{\Gamma}, \quad \mathcal{C} := \frac{a_1\, N_2\, \Delta}{\Gamma} + a_0\, N_2.$$
(13)

When $k_2(\beta - d_2) > a_2 d_2$ the logistic growth of n_2 at N_2 is positive, which implies $\mathcal{C} > 0$ and hence the discriminant $\mathcal{D} = \mathcal{B}^2 + 4\mathcal{A}\mathcal{C} > \mathcal{B}^2$. Since the product of the two roots equals $-\mathcal{C}/\mathcal{A} < 0$, exactly one root is positive:

$$n_1^* = \frac{\mathcal{B} + \sqrt{\mathcal{D}}}{2\mathcal{A}}. \tag{14}$$

The coexistence equilibrium is

$$E_2 = \left(\frac{\mathcal{B} + \sqrt{\mathcal{D}}}{2\mathcal{A}},\ \frac{a_2 d_2}{\beta - d_2},\ 0,\ \frac{a_0 n_1^* + \Delta}{\Gamma}\right), \tag{15}$$

and is biologically admissible when: (i) $\beta > d_2$; (ii) $k_2(\beta - d_2) > a_2 d_2$; and (iii) $a_0 n_1^* + \Delta > 0$ (ensures $p_2^* > 0$).

V. BASIC REPRODUCTION NUMBER

5.1 Derivation via the Next-Generation Matrix

Linearising the predator subsystem at $E_1 = (n_1^*, n_2^*, 0, 0)$ with $b = \beta n_2^*/(a_2 + n_2^*) > 0$:

$$\begin{aligned} \dot{p}_1 &= b p_2(t) - b p_2(t - \tau) - d_1 p_1, \\ \dot{p}_2 &= b p_2(t - \tau) - d_2 p_2. \end{aligned} \tag{16) (17}$$

Following van den Driessche and Watmough [13], new individuals enter p_1 at rate $b\,p_2$. A juvenile born at time s survives the maturation interval $[s, s + \tau]$ with constant death rate d_1, reaching adulthood with probability $\Pi(\tau) = \exp\left(-\int_0^\tau d_1\, ds'\right) = e^{-d_1 \tau}$.

The matrices are

$$F = \begin{pmatrix} 0 & b \\ 0 & b e^{-d_1 \tau} \end{pmatrix}, \qquad V = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}. \tag{18}$$

The characteristic polynomial of $K = FV^{-1}$ is $\lambda\left(\lambda - b e^{-d_1 \tau}/d_2\right) = 0$, giving

$$\mathcal{R}_0 = \rho(K) = \frac{\beta\, n_2^*\, e^{-d_1 \tau}}{d_2(a_2 + n_2^*)}. \tag{19}$$

5.2 Biological Interpretation

$\mathcal{R}_0$ can be read as a product of three factors: (i) the per-capita rate at which prey are converted to juvenile predators under Holling II saturation, $\beta n_2^*/(a_2 + n_2^*)$; (ii) the probability $e^{-d_1 \tau}$ that a juvenile survives the entire maturation period; and (iii) the expected adult lifespan $1/d_2$. Because $\mathcal{R}_0$ decreases strictly with τ, extending the maturation period always reduces the capacity of predators to establish themselves.

Theorem 4 (Predator invasion threshold). Let $\mathcal{R}_0$ be as in ((19)). (i) If $\mathcal{R}_0 < 1$, any small predator perturbation at E_1 decays. (ii) If $\mathcal{R}_0 > 1$, the perturbation grows and predators invade. (iii) $\mathcal{R}_0 = 1$ is the transcritical bifurcation point.

VI. SENSITIVITY ANALYSIS OF $\mathcal{R}_0$

Definition 2 (Normalised sensitivity index). $\Upsilon_\theta^{\mathcal{R}_0} = (\partial \mathcal{R}_0/\,\partial \theta) \cdot (\theta/\mathcal{R}_0)$.

Direct differentiation of ((19)) yields the indices for parameters that appear explicitly in $\mathcal{R}_0$:

$$\begin{aligned} \Upsilon_\beta^{\mathcal{R}_0} &= +1, & \Upsilon_{d_2}^{\mathcal{R}_0} &= -1, \\ \Upsilon_{d_1}^{\mathcal{R}_0} &= -d_1 \tau, & \Upsilon_\tau^{\mathcal{R}_0} &= -d_1 \tau, \\ \Upsilon_{a_2}^{\mathcal{R}_0} &= -\frac{a_2}{a_2 + n_2^*} \in (-1,0), & \Upsilon_{n_2^*}^{\mathcal{R}_0} &= +\frac{a_2}{a_2 + n_2^*} \in (0,1). \end{aligned}$$

Prey parameters $\theta \in \{r_1, r_2, k_1, k_2, a_0\}$ influence $\mathcal{R}_0$ indirectly through n_2^*; their indices satisfy $\Upsilon_\theta^{\mathcal{R}_0} = \Upsilon_{n_2^*}^{\mathcal{R}_0} \cdot \Upsilon_\theta^{n_2^*}$, all of which are positive. The fear coefficient a_1 and the maximum predation rate α have no first-order effect on $\mathcal{R}_0$.

Remark 1. (i) $\Upsilon_\beta^{\mathcal{R}_0} = \left|\Upsilon_{d_2}^{\mathcal{R}_0}\right| = 1$: these are the primary management parameters. (ii) $\Upsilon_{d_1}^{\mathcal{R}_0} = \Upsilon_\tau^{\mathcal{R}_0} = -d_1\tau$: both enter $\mathcal{R}_0$ only through the product $d_1\tau$, the expected juvenile mortality during maturation. (iii) $\Upsilon_{a_2}^{\mathcal{R}_0} + \Upsilon_{n_2^*}^{\mathcal{R}_0} = 0$: an equal proportional increase in a_2 and n_2^* leaves $\mathcal{R}_0$ unchanged.

Normalised sensitivity indices of $\mathcal{R}_0$ for System (1), in decreasing order of $|\Upsilon|$.

Parameter	Role	Index	Sign
β	Conversion efficiency	$+1$	$+$
d_2	Adult predator mortality	-1	$-$
d_1	Juvenile predator mortality	$-d_1\tau$	$-$
τ	Maturation delay	$-d_1\tau$	$-$
a_2	Half-saturation constant	$-a_2/(a_2+n_2^*)$	$-$
n_2^*	Prey density, predation zone	$+a_2/(a_2+n_2^*)$	$+$
r_2, k_2	Prey growth/capacity, zone	positive, chain rule	$+$
r_1, k_1	Prey growth/capacity, refuge	positive, chain rule	$+$
$a_1,\ \alpha$	Fear coefficient; predation rate	0	—

:

VII. LOCAL STABILITY ANALYSIS

For the DDE system linearised about an equilibrium x^*, solutions of the form $e^{\lambda t}v$ exist when $\det(\lambda I - J - J_\tau e^{-\lambda\tau}) = 0$, where $J = D_1F(x^*, x^*)$ and $J_\tau = D_2F(x^*, x^*)$. Because the only delayed term in System ((1)) is $p_2(t-\tau)$, appearing in f_3 and f_4, the matrix J_τ has non-zero entries only in column 4, with $b := \frac{\beta n_2^*}{a_2+n_2^*}$:

$$J_\tau = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\dfrac{\beta n_2^*}{a_2+n_2^*} \\ 0 & 0 & 0 & \dfrac{\beta n_2^*}{a_2+n_2^*} \end{pmatrix} \tag{20}$$

where n_2^* takes the value at the equilibrium under consideration.

7.1 Local Stability of E_0

At $E_0 = (0,0,0,0)$ all state variables vanish, which forces $b = 0$ and hence $J_\tau = 0$. The current-time Jacobian simplifies to:

$$J|_{E_0} = \begin{pmatrix} r_1 - a_0 & a_0 & 0 & 0 \\ a_0 & r_2 - a_0 & 0 & 0 \\ 0 & 0 & -d_1 & 0 \\ 0 & 0 & 0 & -d_2 \end{pmatrix} \tag{21}$$

The characteristic equation factors immediately: the predator block yields eigenvalues $\lambda = -d_1 < 0$ and $\lambda = -d_2 < 0$, while the prey block $A_0 = \begin{pmatrix} r_1 - a_0 & a_0 \\ a_0 & r_2 - a_0 \end{pmatrix}$ contributes two eigenvalues whose product is $\det(A_0) = r_1 r_2 - a_0(r_1 + r_2)$.

Theorem 5 (Local stability of E_0). The trivial equilibrium $E_0 = (0,0,0,0)$ is unstable whenever $\max(r_1, r_2) > a_0$. In particular, under the existence condition $a_0(r_1 + r_2) > r_1 r_2$ for E_1, the prey block A_0 has negative determinant, so E_0 is a saddle point.

Proof. Since $d_1, d_2 > 0$, the predator eigenvalues $-d_1$ and $-d_2$ are always negative. The eigenvalues of A_0 are $\max(r_1, r_2) - a_0$ and $\min(r_1, r_2) - a_0$. Whenever $\max(r_1, r_2) > a_0$, the larger eigenvalue is positive, so E_0 is unstable. Under $a_0(r_1+r_2) > r_1 r_2$, $\det(A_0) < 0$, confirming eigenvalues of opposite sign and a saddle structure. Biologically, total extinction is unstable whenever intrinsic prey growth exceeds the migration drain.

7.2 Local Stability of $\boldsymbol{E_1}$

Theorem 6 (Local stability of $\boldsymbol{E_1}$). $E_1 = (n_1^*, n_2^*, 0, 0)$ is locally asymptotically stable if and only if (S1) $\mathcal{R}_0 < 1$; (S2) $r_1(1-2n_1^*/k_1) + r_2(1-2n_2^*/k_2) < 2a_0$; (S3) $[r_1(1-2n_1^*/k_1) - a_0][r_2(1-2n_2^*/k_2) - a_0] > a_0^2$.

Proof. Set $a_{11} := r_1(1-2n_1^*/k_1) - a_0$ and $a_{22} := r_2(1-2n_2^*/k_2) - a_0$. The current-time and delayed Jacobians at E_1 are:

$$J|_{E_1} = \begin{pmatrix} a_{11} & a_0 & 0 & a_1 n_2^* \\ a_0 & a_{22} & 0 & -a_1 n_2^* - \dfrac{\alpha n_2^*}{a_2 + n_2^*} \\ 0 & 0 & -d_1 & b \\ 0 & 0 & 0 & -d_2 \end{pmatrix}, \quad J_\tau|_{E_1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -b \\ 0 & 0 & 0 & b \end{pmatrix}. \tag{22}$$

The characteristic matrix $(\lambda) = \lambda I - J|_{E_1} - J_\tau|_{E_1} e^{-\lambda\tau}$ is:

$$M(\lambda) = \begin{pmatrix} \lambda - a_{11} & -a_0 & 0 & -a_1 n_2^* \\ -a_0 & \lambda - a_{22} & 0 & a_1 n_2^* + \dfrac{\alpha n_2^*}{a_2 + n_2^*} \\ 0 & 0 & \lambda + d_1 & -b\left(1 - e^{-\lambda\tau}\right) \\ 0 & 0 & 0 & \lambda + d_2 - b\,e^{-\lambda\tau} \end{pmatrix}. \tag{23}$$

Expanding $\det M(\lambda)$ along column 3, the characteristic equation at E_1 factors as

$$(\lambda + d_1)\left(\lambda + d_2 - be^{-\lambda\tau}\right)\left[\lambda^2 - (a_{11} + a_{22})\lambda + \left(a_{11}a_{22} - a_0^2\right)\right] = 0.$$

(24)

Factor 1 gives $\lambda = -d_1 < 0$. All roots of the factor $h_2(\lambda) = \lambda + d_2 - be^{-\lambda\tau}$ lie in the left half-plane iff $b < d_2$, which is $\mathcal{R}_0 < 1$ (condition S1). The quadratic factor satisfies Routh-Hurwitz iff trace <0 (condition S2) and determinant >0 (condition S3).

7.3 Local Stability of E_2

At E_2, the equilibrium identity $\beta n_2^*/(a_2 + n_2^*) = d_2$ holds, so $b = d_2$. The Jacobian entries are: $b_{11} := r_1(1-2n_1^*/k_1) - a_0$, $b_{12} := a_1 p_2^* + a_0$, $b_{14} := a_1 n_2^*$, $b_{21} := a_0$, $b_{22} := r_2(1-2n_2^*/k_2) - (a_1 p_2^* + a_0) - \alpha a_2 p_2^*/(a_2 + n_2^*)^2$, $b_{24} := -a_1 n_2^* - \alpha d_2/\beta$

$b_{42} := \beta p_2^* a_2/(a_2 + n_2^*)^2 > 0$. The current-time and delayed Jacobians at E_2 are:

$$J|_{E_2} = \begin{pmatrix} b_{11} & b_{12} & 0 & b_{14} \\ b_{21} & b_{22} & 0 & b_{24} \\ 0 & 0 & -d_1 & d_2 \\ 0 & b_{42} & 0 & -d_2 \end{pmatrix}, \quad J_\tau|_{E_2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -d_2 \\ 0 & 0 & 0 & d_2 \end{pmatrix}. \tag{25}$$

The characteristic matrix $M(\lambda) = \lambda I - J|_{E_2} - J_\tau|_{E_2} e^{-\lambda\tau}$ is:

$$M(\lambda) = \begin{pmatrix} \lambda - b_{11} & -b_{12} & 0 & -b_{14} \\ -b_{21} & \lambda - b_{22} & 0 & -b_{24} \\ 0 & 0 & \lambda + d_1 & -d_2\left(1 - e^{-\lambda\tau}\right) \\ 0 & -b_{42} & 0 & \lambda + d_2 - d_2 e^{-\lambda\tau} \end{pmatrix}. \tag{26}$$

Expanding along column 3 gives $\det M(\lambda) = (\lambda + d_1) \cdot \det M_{33}(\lambda)$, where:

$$M_{33}(\lambda) = \begin{pmatrix} \lambda - b_{11} & -b_{12} & -b_{14} \\ -b_{21} & \lambda - b_{22} & -b_{24} \\ 0 & -b_{42} & \lambda + d_2 - d_2 e^{-\lambda\tau} \end{pmatrix}. \tag{27}$$

Theorem 7 (Local stability of E_2). Assume $\mathcal{R}_0 > 1$ so that E_2 exists. (i) At $\tau = 0$, E_2 is locally asymptotically stable iff the Routh-Hurwitz conditions

(H1) $b_{11} + b_{22} < 0,$
(H2) $b_{11}b_{22} - b_{12}b_{21} > b_{42}b_{24},$
(H3) $b_{42}(b_{24}b_{11} - b_{14}b_{21}) > 0,$
(H4) $(-b_{11} - b_{22})(b_{11}b_{22} - b_{12}b_{21} - b_{42}b_{24}) > b_{42}(b_{24}b_{11} - b_{14}b_{21})$

hold simultaneously. (ii) For $\tau \in (0, \tau_H)$, E_2 remains locally asymptotically stable under (H1)–(H4). (iii) At $\tau = \tau_H$ a Hopf bifurcation occurs; for $\tau > \tau_H$, E_2 is unstable.

Proof. Expanding $\det M_{33}(\lambda)$ along its last row gives

$$\det M_{33} = \left(\lambda + d_2 - d_2 e^{-\lambda\tau}\right)P_2(\lambda) + b_{42}[-b_{24}(\lambda - b_{11}) - b_{14}b_{21}],$$

Where,
$P_2(\lambda) = \lambda^2 - (b_{11} + b_{22})\lambda + (b_{11}b_{22} - b_{12}b_{21})$.

At $\tau = 0$ this reduces to the cubic $\lambda^3 - \mathcal{T}\lambda^2 + \mathcal{Q}\lambda + \mathcal{R} = 0$ with $\mathcal{T} := b_{11} + b_{22}$, $\mathcal{Q} := b_{11}b_{22} - b_{12}b_{21} - b_{42}b_{24}$, $\mathcal{R} := b_{42}(b_{24}b_{11} - b_{14}b_{21})$. The Routh-Hurwitz criterion for the cubic gives conditions (H1)–(H4). Continuity of eigenvalues in τ gives part (ii); the first imaginary-axis crossing at τ_H gives part (iii).

VIII. GLOBAL STABILITY ANALYSIS

Theorem 8 (Global Stability of E_0). The trivial equilibrium $E_0 = (0,0,0,0)$ of System (4.1) is globally asymptotically stable if $r_1 < 0$, $r_2 < 0$ and $\beta \leq \alpha$.

Biologically, total extinction becomes the global outcome only when both prey populations are intrinsically declining ($r_1, r_2 < 0$): in that case neither patch can sustain positive growth, movement between patches merely redistributes an ever-shrinking population, and the predator, denied a viable prey base and constrained by $\beta \leq \alpha$, eventually disappears. Under the standard assumption $r_1, r_2 > 0$, the equilibrium E_0 is a saddle point and the long-run outcome is determined by $\mathcal{R}_0$, specifically whether the system is attracted to E_1 or E_2.

Theorem 9 (Global stability of E_1). E_1 is globally asymptotically stable in $\mathbb{R}^4_+$ provided $\mathcal{R}_0 < 1$, conditions (S2)–(S3) hold, and $\beta \leq d_2(1 + a_2/n_2^*)$

Proof. Define

$$V(t) = 1/2\,(n_1 - n_1^*)^2 + 1/2\,(n_2 - n_2^*)^2 + p_1 + p_2 + \int_{t-\tau}^{t} \frac{\beta n_2(s) p_2(s)}{a_2 + n_2(s)}\,ds.$$

(28)

The integral term cancels the delayed terms in $\dot{V}$ arising from the predator equations. Under (S2)–(S3) the prey contribution is negative definite. Using $\mathcal{R}_0 < 1$ and the inequality $n_2/(a_2 + n_2) \leq n_2^*/(a_2 + n_2^*) + \varepsilon$ near n_2^*:

$$\dot{V} \leq -\lambda_1(n_1 - n_1^*)^2 - \lambda_2(n_2 - n_2^*)^2 - d_1 p_1 - \delta p_2 \leq 0$$

(29)

for some $\lambda_i, \delta > 0$. LaSalle's invariance principle [14] gives convergence to $\{E_1\}$.

Theorem 10 (Global stability of E_2). Let $\mathcal{R}_0 > 1$. E_2 is globally asymptotically stable in the interior of $\mathbb{R}^4_+$ provided: (i) $\beta > d_2$ and $k_2(\beta - d_2) > a_2 d_2$; (ii) $r_1/k_1 + r_2/k_2 > \max\{\alpha p_2^*/(a_2 + n_2^*)^2, a_1 n_2^*/n_1^*\}$; (iii) $\tau < \tau_{\max} = \min\{1/d_1, 1/d_2, (a_2 + n_2^*)/(\beta n_2^*)\}$; (iv) $a_0 > \frac{1}{2}\left(a_1 p_2^* + \alpha p_2^*/(a_2 + n_2^*)\right)$.

Proof. Define the Lyapunov-Krasovskii functional

$$\begin{aligned} V(t) &= c_1(n_1 - n_1^* - n_1^* \ln n_1/n_1^*) + c_2(n_2 - n_2^* - n_2^* \ln n_2/n_2^*) \\ &\quad + c_3(p_1 - p_1^* - p_1^* \ln p_1/p_1^*) + c_4(p_2 - p_2^* - p_2^* \ln p_2/p_2^*) + V_2(t), \end{aligned}$$

Where

$$V_2(t) = \int_{t-\tau}^{t} \left[\frac{\beta n_2 p_2}{a_2 + n_2} - \frac{\beta n_2^* p_2^*}{a_2 + n_2^*} - \frac{\beta n_2^* p_2^*}{a_2 + n_2^*} \ln \frac{\beta n_2 p_2 (a_2 + n_2^*)}{\beta n_2^* p_2^* (a_2 + n_2)}\right] ds,$$

(30)

with $c_1 = c_2 = 1$ and $c_3 = c_4 = \alpha/\beta$. The derivative of V_2 cancels all delayed terms in $\dot{V}_1$. Under the stated conditions, $\dot{V} \leq -W(n_1, n_2, p_1, p_2) \leq 0$ where

$$W = \lambda_1(n_1 - n_1^*)^2 + \lambda_2(n_2 - n_2^*)^2 + \lambda_3(p_1 - p_1^*)^2 + \lambda_4(p_2 - p_2^*)^2 + \lambda_5(n_2/a_2 + n_2 - n_2^*/a_2 + n_2^*)^2$$

(31)

is positive definite. LaSalle's principle [14] gives convergence to E_2.

IX. BIFURCATION ANALYSIS

9.1 Transcritical Bifurcation at $\mathcal{R}_0 = 1$

Taking β as the bifurcation parameter, the critical value is $\beta^* = d_2(1 + a_2/n_2^*)e^{d_1\tau}$.

Theorem 11 (Transcritical bifurcation). At $\beta = \beta^*$ (i) E_1 is non-hyperbolic with a simple zero eigenvalue; (ii) for $\mathcal{R}_0 < 1$, E_1 is stable and E_2 is infeasible; (iii) for $\mathcal{R}_0 > 1$, E_1 loses stability and E_2 emerges. The transversality condition $d\lambda/d\beta|_{\beta=\beta^*} = [n_2^*/(a_2 + n_2^*)]/(1 + d_2\tau) > 0$ holds.

Proof. Setting λ=0 in the predator characteristic factor gives $d_2 = bn_2^*/(a_2 + n_2^*)$, which is exactly $\beta = \beta^*$. Implicit differentiation at λ=0 yields the stated transversality inequality.

9.2 Hopf Bifurcation Induced by the Maturation Delay

Substituting $\lambda = i\omega$ $(\omega > 0)$ into the characteristic equation of E_2, separating real and imaginary parts, and eliminating τ leads to the frequency polynomial

$$h(z) := z^3 + c_1z^2 + c_2z + c_3 = 0, \qquad z = \omega^2,$$

(32)

with coefficients c_1, c_2, c_3 determined by the Jacobian entries at E_2. When h admits a positive real root $z_0 = \omega_0^2$, the critical delay is

$$\tau_j = \frac{1}{\omega_0}\arccos\left(\frac{(A_0-\omega_0^2)(v_3-v_1\omega_0^2)+A_1\omega_0(v_2\omega_0-\omega_0^3)}{d_2\left[(A_0-\omega_0^2)^2+A_1^2\omega_0^2\right]}\right)+\frac{2j\pi}{\omega_0}, \quad j = 0,1,2,\ldots,$$

(33)

and $\tau_0^* = \min_{j\geq 0}\{\tau_j : \tau_j > 0\}$.

Theorem 12 (Hopf bifurcation). Assume (H1)–(H4) hold and h has a positive real root z_0. (i) E_2 is locally asymptotically stable for $\tau \in [0, \tau_0^*)$. (ii) At $\tau = \tau_0^*$a Hopf bifurcation occurs and a branch of periodic orbits is born. (iii) The transversality condition $\text{sign}\{\text{Re}[d\lambda/d\tau]_{\tau=\tau_0^*}\} = \text{sign}\{h'(z_0)\}$ holds. If h has no positive real root, E_2 is delay-independently stable.

X. CONCLUSION

A two-patch delay differential equation predator-prey model incorporating density-dependent fear-driven prey movement and a stage-structured predator maturation delay has been developed and fully analysed. Exponential boundedness is established through the explicit bound $V(t) \leq C_0e^{\mu t}$, $\mu = \max\{r_1, r_2\}$ which follows from Gronwall's inequality after the migration and delay flux terms cancel exactly. The predator-free equilibrium E_1 exists whenever $a_0(r_1 + r_2) > r_1r_2$, a condition confirmed by applying the Intermediate Value Theorem to a cubic polynomial, and the coexistence equilibrium E_2 is expressed in the explicit closed form given in ((15)).

The basic reproduction number $\mathcal{R}_0 = \beta n_2^* e^{-d_1\tau}/[d_2(a_2 + n_2^*)]$ is strictly decreasing in τ: each additional unit of maturation time reduces $\mathcal{R}_0$ by the factor e^{-d_1}. The sensitivity analysis identifies β and d_2 as the primary management levers $|\Upsilon| = 1$, the product $d_1\tau$ as a secondary control $|\Upsilon| = d_1\tau$, and confirms that the fear coefficient a_1 and maximum predation rate α have no first-order influence on $\mathcal{R}_0$.

Global asymptotic stability of E_1 when $\mathcal{R}_0 < 1$ and of E_2 when $\mathcal{R}_0 > 1$ (subject to $\tau < \tau_{\max}$) is proved using explicit Lyapunov-Krasovskii functionals. The maturation delay τ drives a Hopf bifurcation of E_2 at the threshold τ_0^* obtained from the scalar cubic ((32)); for $\tau > \tau_0^*$ the coexistence equilibrium becomes unstable and persistent predator-prey oscillations emerge.

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