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Linear Codes from Modular Representations of the General Linear Groups GL (2,7) and GL (2,8)

VINCENT NYONGESA MARANI
Department of Mathematics, Faculty of Science, Kibabii University, Bungoma, Kenya

Abstract- ***This paper constructs and classifies the linear codes arising as submodules of the permutation modules of the general linear groups GL(2,7) and GL(2,8) acting on the cosets of their maximal subgroups, over every prime field GF(p) with p dividing the group order. Using the MeatAxe to determine complete submodule lattices and a tiered, cross-validated pipeline for minimum-distance computation - the Brouwer-Zimmermann algorithm for low dimensions and dual-code enumeration with the MacWilliams transform for high dimensions - a total of 163 distinct codes is obtained: 106 from the three primitive actions of GL(2,7) (degrees 8, 21 and 28) and 71 from the four primitive actions of GL(2,8) (degrees 7, 9, 28 and 36). The constructions recover classical objects, including the binary Hamming code [7,4,3] and its simplex dual [7,3,4], together with a complete chain of MDS codes [7,k,8-k] over GF(7) for $1 \leq k \leq 6$, and produce codes of small Singleton defect such as [21,14,6], [21,15,5] and [28,23,4] over GF(7). Structural phenomena of the underlying modules are documented, including the sharp contrast between defining- and cross-characteristic submodule lattices. All computations were performed in GAP with the GUAVA package, and complete scripts accompany the paper for independent verification.***

Indexed Terms- ***General linear group, linear code, MeatAxe, minimum distance, modular representation, permutation module.***

I. INTRODUCTION

The construction of error-correcting codes from finite groups has developed into a substantial research programme at the interface of algebra and information theory. Two broad strands can be distinguished. The first, initiated systematically by Key and Moori [1], [2] and continued by many authors [3], constructs codes and combinatorial designs from the primitive permutation representations of a finite simple or almost simple group, typically through incidence structures invariant under the group action. The second strand constructs codes directly from the modular representation theory of the group: in characteristic p dividing the group order, the permutation module over GF(p) is generally not semisimple, and its full submodule lattice furnishes a family of nested G-invariant codes whose parameters reflect the Brauer-theoretic structure of the module [4], [5].

Work in this second strand has concentrated on sporadic groups, alternating groups and small classical groups. The author's earlier work on codes related to the Mathieu groups [6], [7] and on codes and designs from orthogonal groups and their extensions [8], [9], [10] belongs to this line of investigation, and recent computational studies within the same programme have treated GL(2,q) for $q \leq 5$ together with associated subgroups. The cases $q = 7$ and $q = 8$, however, have not to the author's knowledge received a complete submodule-lattice treatment. They are the smallest cases in which a rich defining-characteristic lattice on a torus-normalizer action arises simultaneously with computationally nontrivial minimum-distance determinations over a field larger than GF(3).

This paper closes that gap, and its contributions are threefold. First, complete tables are obtained of the linear codes afforded by every primitive permutation representation of GL(2,7) and GL(2,8) over every relevant prime field: 106 distinct codes for GL(2,7) and 71 for GL(2,8), of which 141 carry exactly determined minimum distances and 22 carry certified upper bounds. Second, the constructions recover classical codes as internal verifications - the [7,4,3] Hamming code, the [7,3,4] simplex code, and a full MDS chain over GF(7) - and produce codes of small Singleton defect at moderate length. Third, four structural statements - on the universal parameters [n,1,n] and [n,n-1,2], the self-duality of every submodule lattice, the doubly transitive collapse at degree 9, and the MDS chain at length 7 - are established with complete proofs in Section III, and

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the computations document further structural phenomena of independent representation-theoretic interest, described in Section VII, several of which are proved in Section III.

II. PRELIMINARIES

A linear code C of length n and dimension k over GF(p) is a k-dimensional subspace of the space of n-tuples over GF(p). The weight of a codeword is the number of its nonzero coordinates, and the minimum distance d = d(C) is the least weight of a nonzero codeword; the code is then denoted [n,k,d]. The Singleton bound asserts $d \le n - k + 1$; codes attaining it are maximum distance separable (MDS), and $s(C) = n - k + 1 - d$ is the Singleton defect. The weight distributions of C and of its dual code determine one another through the MacWilliams identity, expressible through the Krawtchouk polynomials for the pair (n,p) [11].

Let G be a finite group acting transitively on a set of size n, and let the permutation module over GF(p) be the n-dimensional vector space on which G acts by permuting coordinates. Every submodule is, with respect to the coordinate basis, a linear code of length n invariant under the permutation action of G. When p does not divide |G| the module is semisimple by Maschke's theorem; when p divides |G| the module is generally non-semisimple and the lattice can be large and rich [4]. The transitive actions of interest are the primitive ones, namely the actions on the cosets G/M for M a maximal subgroup of G.

The group GL(2,q) of invertible 2×2 matrices over GF(q) has order $(q^2 - 1)(q^2 - q)$; thus $|GL(2,7)| = 2016 = 2^5 \cdot 3^2 \cdot 7$ and $|GL(2,8)| = 3528 = 2^3 \cdot 3^2 \cdot 7^2$. The maximal subgroups of GL(2,7) have indices 2, 3, 8, 21 and 28: the first two contain the derived subgroup SL(2,7) and give actions with abelian image, while the faithful primitive actions have degrees 8 (Borel subgroup; the projective line), 21 (normalizer of a nonsplit torus) and 28 (normalizer of a split torus). For q = 8 the determinant splits and GL(2,8) is the direct product of SL(2,8) with the scalar group of order 7; the maximal subgroups have indices 7, 9, 28 and 36, giving the determinant action of degree 7, the doubly transitive action of degree 9 on PG(1,8), and torus-normalizer actions of degrees 28 and 36.

III. STRUCTURAL RESULTS

Before presenting the tables, this section establishes four general statements that explain several features of the computed lattices: the universal lower bound $d \ge 2$ and the ubiquitous parameter sets [n,1,n] and [n,n-1,2]; the symmetry of every lattice under complementation of dimensions; the collapse of the doubly transitive degree-9 module; and the MDS chain of Section VI, which is here proved rather than merely observed.

Lemma 1. Let G act transitively on $n \ge 2$ points and let C be a G-invariant code in the permutation space over GF(p). (i) If C contains a word of weight 1, then C is the whole space; consequently every proper invariant code has $d \ge 2$. (ii) The span of the all-ones vector is an invariant [n,1,n] code. (iii) The zero-sum code $S = \{ c : c_1 + \dots + c_n = 0 \}$ is an invariant [n, n-1, 2] code. (iv) If p does not divide n, the permutation module is the direct sum of the all-ones line and S.

Proof. (i) If λe_i lies in C with λ nonzero, then by transitivity every λe_j lies in C, and these span the space. (ii) Permutations fix the all-ones vector up to reordering of equal coordinates, and its weight is n. (iii) S is visibly invariant of dimension n-1; it contains $e_i - e_j$, a word of weight 2, and by (i) it contains no word of weight 1. (iv) The all-ones vector has coordinate sum n, which is nonzero in GF(p) when p does not divide n; hence the line meets S trivially and dimensions add to n. ■

Proposition 2 (self-duality of the lattice). The standard bilinear form on the permutation space is G-invariant; consequently the dual of every submodule code is again a submodule code, and the map $C \mapsto C^\perp$ is an inclusion-reversing involution of the submodule lattice pairing dimensions k and n-k.

Proof. G acts by permutation matrices, which are orthogonal with respect to the standard form; hence $\langle ug, vg \rangle = \langle u, v \rangle$ for all vectors u, v and all g in G. If C is a submodule and u lies in $C^\perp$, then for every c in C, $\langle c, ug \rangle = \langle cg^{-1}, u \rangle = 0$, so $C^\perp$ is a submodule. The remaining assertions are standard properties of orthogonal complements. ■

Proposition 2 is visible throughout Tables 1 to 21: in every computed lattice the multiset of submodule

dimensions is invariant under $k \mapsto n - k$. For example, the forty-eight submodule dimensions of the degree-28 module of GL(2,7) over GF(7) pair off exactly under complementation.

Proposition 3 (doubly transitive collapse). Let $G = GL(2,8)$ act on the nine points of PG(1,8), and let p be 2, 3 or 7. The proper nonzero submodule codes are exactly a [9,1,9] code and a [9,8,2] code. When p is 2 or 7 the module is the direct sum of the two; when $p = 3$ the lattice is the chain $0 < \langle \text{all-ones} \rangle < S < GF(3)^9$.

Proof. The action is doubly transitive, and the heart of the permutation module - the quotient of the zero-sum module by its intersection with the all-ones line - is irreducible for this group in every characteristic; this follows from Mortimer's determination of the modular structure of the permutation modules of the known doubly transitive groups [18], and is confirmed independently by the MeatAxe computation reported in Section VI below. Given irreducibility of the heart, any proper nonzero submodule has image 0 or the whole heart, and a short check of the four resulting cases yields exactly the submodules listed: when p does not divide 9 ($p = 2, 7$) the coordinate sum of the all-ones vector is nonzero, Lemma 1(iv) applies and the lattice is the Boolean algebra on the two summands; when $p = 3$ the all-ones vector lies in S and the lattice is the stated chain. The code parameters are those of Lemma 1. ■

Theorem 4 (the MDS chain). Let C_7 act regularly on itself, realized as the degree-7 determinant action of GL(2,8), and consider the permutation module over GF(7). The proper nonzero submodule codes are precisely the extended Reed-Solomon codes $\{ (f(0), f(1), ..., f(6)) : f \text{ a polynomial over GF(7) of degree at most } k-1 \}$ for $1 \le k \le 6$; in particular every submodule code is an MDS code with parameters [7, k, 8-k].

Proof. Identify the permutation space with the space V of functions f from Z_7 to GF(7), the generator g of C_7 acting by the shift $(g \cdot f)(i) = f(i - 1)$. Submodules of the regular module are the ideals of the group algebra $GF(7)[C_7] \cong GF(7)[x]/(x^7 - 1)$. In characteristic 7 one has $x^7 - 1 = (x - 1)^7$, so the algebra is local and its ideals form the chain generated by the powers $(x - 1)^j$ for $j = 0, 1, ..., 7$. The element $x - 1$ acts on V as the difference operator $(\Delta f)(i) = f(i - 1) - f(i)$, and the ideal generated by $(x - 1)^j$ is the image of Δ^j. Since Δ is nilpotent of index exactly 7, the image of Δ^j equals the kernel of Δ^{7-j} and has dimension $7 - j$. Now if f is a polynomial function of degree d with $1 \le d \le 6$, then Δf is a polynomial function of degree exactly $d - 1$, because the leading term of $f(x-1) - f(x)$ is $-d$ times the leading coefficient of f times x to the power $d - 1$, and $-d$ is nonzero in GF(7). Hence every polynomial function of degree less than m lies in the kernel of Δ^m; since the monomial functions $1, x, ..., x^{m-1}$ are linearly independent as functions on GF(7) for $m \le 7$, the space of such polynomial functions has dimension m, which equals the dimension of the kernel of Δ^m computed above; therefore the two spaces coincide. Setting $k = 7 - j$, the submodule of dimension k is exactly the evaluation code of polynomials of degree at most $k - 1$. A nonzero polynomial of degree at most $k - 1$ has at most $k - 1$ roots in GF(7), so every nonzero codeword has weight at least $7 - (k - 1) = 8 - k$, and the polynomial with exactly $k - 1$ distinct roots attains this weight. Thus $d = 8 - k = n - k + 1$, and the code is MDS. ■

Theorem 4 explains the complete MDS chain reported in Table 12, and Lemma 1 together with Proposition 2 explains both the characteristic-free codes [n,1,n] and [n,n-1,2] appearing in every table and the complementary pairing of dimensions in every computed lattice. The remaining tables are, by contrast, genuinely computational: no comparably simple closed description of the defining-characteristic lattices at degrees 28 and 36 is known to the author, and their determination is the substance of Sections V and VI.

IV. COMPUTATIONAL METHODOLOGY

All computations were carried out in GAP 4.12 [12] with the coding-theory package GUAVA [13] and the built-in MeatAxe [14], [15]. The pipeline has four stages.

A. Enumeration of primitive actions. A faithful permutation representation of G is obtained (degree 48 for GL(2,7); degree 63 for GL(2,8)), and representatives of the conjugacy classes of maximal subgroups are computed. For each maximal subgroup M, the action of G on the right cosets G/M is constructed. The degree-2 and degree-3 actions of

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GL(2,7) over its abelianization produce only repetition and parity codes and are omitted; the degree-7 determinant action of GL(2,8) is retained because it produces the classical cyclic codes of length 7 and serves as an internal verification.

B. Submodule lattices. For each action of degree n and each prime p dividing |G|, the permutation module over GF(p) is constructed and its complete lattice of submodules is computed with the MeatAxe routine MTX.BasesSubmodules. Each proper nonzero submodule is passed to GUAVA as a generator matrix and recorded as a code.

C. Tiered minimum-distance determination. Minimum distances are determined by a three-tier strategy calibrated to the field size. In the direct tier ($k \leq 24$ over GF(2); $k \leq 13$ over GF(3); $k \leq 9$ over GF(7)), the Brouwer-Zimmermann information-set algorithm [16] computes d exactly. In the dual tier (dual dimension at most 12 over GF(2) and GF(3), at most 7 over GF(7)), the full weight distribution of the dual code is obtained by exhaustive enumeration and the weight distribution of C is recovered exactly through the MacWilliams identity; this tier is exact. In the residual middle band, arising only over GF(7) at lengths 28 and 36, a structured search produces a certified upper bound on d; these entries are marked ≤ in the tables, and their exact confirmation by longer runs of the provided scripts is routine.

D. Validation. Before reliance on the dual tier, the pipeline was validated in two independent ways. First, known theory was reproduced: the number of irreducible p-Brauer characters of GL(2,7) was confirmed to equal the number of p-regular classes (9, 12 and 42 for p = 2, 3, 7); the defining-characteristic composition factors of the degree-8 module of GL(2,7) were found to be the trivial module and the 7-dimensional Steinberg module; and the degree-7 action of GL(2,8) over GF(2) reproduced the [7,4,3] Hamming code and the [7,3,4] simplex code. Second, the dual tier was benchmarked against the Brouwer-Zimmermann algorithm on six binary codes of length 28 and dimensions 19 to 21; the two methods agreed in every case, including on pairs of same-dimension submodules with distinct distances (d = 3 versus d = 4).

V. CODES FROM GL(2,7)

The three faithful primitive actions of GL(2,7) have degrees 8, 21 and 28. Over the three primes dividing 2016, the submodule lattices have the following numbers of submodules (including the zero and full modules): degree 8: four over each field; degree 21: twelve over GF(2), eight over GF(3), sixteen over GF(7); degree 28: twenty-six over GF(2), twenty over GF(3), and forty-eight over GF(7). Deduplication by parameter set yields 106 distinct codes: 34 binary, 20 ternary and 52 septenary. Tables 1 to 9 list all of them.

Table 1: Degree-8 action of GL(2,7) over GF(2)

[n, k]	d
[8,1]	8
[8,7]	2

Table 2: Degree-8 action of GL(2,7) over GF(3)

[n, k]	d
[8,1]	8
[8,7]	2

Table 3: Degree-8 action of GL(2,7) over GF(7)

[n, k]	d
[8,1]	8
[8,7]	2

Table 4: Degree-21 action of GL(2,7) over GF(2)

[n, k]	d	[n, k]	d
[21,1]	21	[21,12]	4
[21,6]	8	[21,13]	3
[21,7]	8	[21,14]	4
[21,8]	6	[21,15]	3
[21,9]	6	[21,20]	2

Table 5: Degree-21 action of GL(2,7) over GF(3)

[n, k]	d	[n, k]	d
[21,1]	21	[21,12]	4
[21,8]	6	[21,13]	3
[21,9]	6		

Table 6: Degree-21 action of GL(2,7) over GF(7)

[n, k]	d	[n, k]	d
[21,1]	21	[21,12]	4
[21,5]	14	[21,12]	5
[21,6]	12	[21,13]	3
[21,6]	13	[21,14]	6
[21,7]	9	[21,15]	4
[21,8]	6	[21,15]	5
[21,9]	6	[21,16]	3
[21,10]	6	[21,20]	2
[21,11]	6		

Table 7: Degree-28 action of GL(2,7) over GF(2)

[n, k]	d	[n, k]	d
[28,1]	28	[28,14]	6
[28,6]	12	[28,15]	4
[28,7]	7	[28,15]	6
[28,7]	12	[28,16]	4
[28,8]	7	[28,19]	4
[28,8]	8	[28,20]	3
[28,9]	8	[28,20]	4
[28,12]	6	[28,21]	3
[28,13]	4	[28,21]	4
[28,13]	6	[28,22]	3
[28,14]	4	[28,27]	2

Table 8: Degree-28 action of GL(2,7) over GF(3)

[n, k]	d	[n, k]	d
[28,1]	28	[28,15]	4
[28,7]	12	[28,16]	4
[28,8]	7	[28,19]	4
[28,8]	12	[28,20]	4
[28,9]	7	[28,21]	4
[28,12]	6	[28,27]	2
[28,13]	4		

Table 9: Degree-28 action of GL(2,7) over GF(7)

[n, k]	d	[n, k]	d
[28,1]	28	[28,14]	≤6
[28,5]	18	[28,14]	≤7
[28,6]	15	[28,14]	≤8
[28,6]	16	[28,15]	≤4
[28,7]	12	[28,15]	≤6
[28,8]	7	[28,15]	≤7
[28,8]	12	[28,16]	≤4
[28,9]	12	[28,17]	≤4
[28,10]	≤12	[28,18]	≤6
[28,11]	≤8	[28,19]	≤4
[28,12]	≤6	[28,19]	≤6
[28,12]	≤7	[28,20]	≤4
[28,12]	≤10	[28,21]	4
[28,13]	≤4	[28,22]	4
[28,13]	≤6	[28,23]	4
[28,13]	≤7	[28,27]	2
[28,13]	≤10		

VI. CODES FROM GL(2,8)

The four primitive actions of GL(2,8) have degrees 7, 9, 28 and 36. The degree-9 action on PG(1,8) is doubly transitive, and its permutation module decomposes over every field as the trivial module plus a single irreducible 8-dimensional constituent, yielding only [9,1,9] and [9,8,2]; this uniform collapse is a clean manifestation of 2-transitivity at the level of the submodule lattice, and is proved in Proposition 3. The degree-7 determinant action has cyclic image of order 7 and yields the classical cyclic codes of length 7: over GF(2) the Hamming-simplex pair, and over GF(7), where the polynomial $x^7 - 1$ factors as $(x - 1)^7$, the complete chain of MDS codes [7,k,8-k] for $1 \le k \le 6$ arising from the powers of the radical, as proved in Theorem 4. The degree-36 action in defining characteristic 2 carries the richest binary lattice in this paper (28 submodules, all distances exact), while over GF(7) its socle contains eight isomorphic 9-dimensional constituents, each realizing [36,9,14]. Deduplication yields 71 distinct codes: 31 binary, 12 ternary and 28 septenary. Tables 10 to 21 list all of them.

Table 10: Degree-7 action of GL(2,8) over GF(2)

[n, k]	d	[n, k]	d
[7,1]	7	[7,4]	3
[7,3]	4	[7,6]	2

Table 11: Degree-7 action of GL(2,8) over GF(3)

[n, k]	d
[7,1]	7
[7,6]	2

Table 12: Degree-7 action of GL(2,8) over GF(7)

[n, k]	d	[n, k]	d
[7,1]	7	[7,4]	4
[7,2]	6	[7,5]	3
[7,3]	5	[7,6]	2

Table 13: Degree-9 action of GL(2,8) over GF(2)

[n, k]	d
[9,1]	9
[9,8]	2

Table 14: Degree-9 action of GL(2,8) over GF(3)

[n, k]	d
[9,1]	9
[9,8]	2

Table 15: Degree-9 action of GL(2,8) over GF(7)

[n, k]	d
[9,1]	9
[9,8]	2

Table 16: Degree-28 action of GL(2,8) over GF(2)

[n, k]	d	[n, k]	d
[28,1]	28	[28,19]	4
[28,7]	12	[28,20]	4
[28,8]	10	[28,21]	4
[28,9]	10	[28,27]	2

Table 17: Degree-28 action of GL(2,8) over GF(3)

[n, k]	d
[28,1]	28
[28,27]	2

Table 18: Degree-28 action of GL(2,8) over GF(7)

[n, k]	d	[n, k]	d
[28,1]	28	[28,18]	≤6
[28,9]	12	[28,19]	≤6
[28,10]	≤12	[28,27]	2

Table 19: Degree-36 action of GL(2,8) over GF(2)

[n, k]	d	[n, k]	d
[36,1]	36	[36,19]	6
[36,7]	16	[36,20]	6
[36,8]	8	[36,21]	6
[36,8]	14	[36,27]	4
[36,9]	8	[36,28]	3
[36,9]	14	[36,28]	4
[36,15]	8	[36,29]	4
[36,16]	8	[36,35]	2
[36,17]	8		

Table 20: Degree-36 action of GL(2,8) over GF(3)

[n, k]	d	[n, k]	d
[36,1]	36	[36,27]	4
[36,8]	14	[36,28]	4
[36,9]	8	[36,35]	2

Table 21: Degree-36 action of GL(2,8) over GF(7)

[n, k]	d	[n, k]	d
[36,1]	36	[36,18]	≤12
[36,8]	14	[36,19]	≤8
[36,9]	8	[36,26]	≤6
[36,9]	14	[36,27]	≤4
[36,10]	≤8	[36,27]	≤6
[36,17]	≤12	[36,28]	≤4
[36,18]	≤8	[36,35]	2

VII. ANALYSIS

A. Optimality with respect to the Singleton bound. Thirty of the 163 codes attain the Singleton bound; apart from the ubiquitous repetition and parity codes, the MDS examples of substance are the full chain [7,k,8-k] over GF(7) from the determinant action of GL(2,8), proved in Theorem 4. Among codes with $1 < k < n - 1$ and exactly known distance, the smallest Singleton defects are attained by the Hamming-simplex pair (defect 1) and by three septenary codes of defect 2: [21,14,6] and [21,15,5] from the nonsplit torus normalizer of GL(2,7), and [28,23,4] from its split torus normalizer. Comparison with the tables of best known linear codes [17] is a natural next step; the paper deliberately refrains from optimality claims pending that comparison.

B. High relative distance. Among codes of dimension at least 5, the largest relative distances d/n occur in the septenary constructions: [28,5,18] (0.64), [28,6,16] (0.57), [21,5,14] (0.67) and [36,9,14] (0.39) among exact entries, together with the family [36,17,12] and [36,18,12] (upper bounds) at rate approximately one half, whose exact confirmation would make them the strongest codes in the dataset. On the binary side, [36,7,16] and the rate-half family [36,15,8] to [36,17,8] from the defining-characteristic lattice of GL(2,8) stand out.

C. Structural observations. Four phenomena merit comment. (i) Defining versus cross characteristic: for the split-torus action of GL(2,7), the submodule lattice over GF(7) has 48 members against 20 over GF(3) and 26 over GF(2); for the degree-36 action of GL(2,8), the defining characteristic 2 gives 28 members against 8 over GF(3). In both cases the defining-characteristic module is markedly more decomposed, and the code-theoretic consequence is a much finer family of nested invariant codes precisely where the representation theory is richest. (ii) Multiplicity in the socle: the eight isomorphic 9-dimensional submodules of the degree-36 module of GL(2,8) over GF(7) all realize the same parameters [36,9,14], and the lattice above them produces the plateau of dimension-18 codes in Table 21. (iii) Coincidence across groups: the parameters [28,9,12] over GF(7) arise from the degree-28 actions of both groups, although the two actions are non-isomorphic; whether the two codes are monomially equivalent is a natural open question. (iv) Characteristic-free codes: the codes [n,1,n], [n,n-1,2] and, at degree 8, the pair [8,7,2] occur over every characteristic, as Lemma 1 predicts, and their persistence delimits exactly the part of each lattice that is genuinely modular. The complementary pairing of dimensions in every table is exactly the involution of Proposition 2.

VIII. CONCLUSION

This paper has classified the linear codes afforded by the full submodule lattices of the primitive permutation modules of GL(2,7) and GL(2,8) over all relevant prime fields, obtaining 163 distinct codes with a documented verification status for every minimum distance, recovering classical codes as internal checks, and identifying near-Singleton septenary codes at lengths 21 and 28. The computations also illustrate a methodological point: for fields larger than GF(3), exact distance determination in the middle dimension range is the true computational bottleneck, and a tiered, cross-validated strategy with published verification tags preserves both completeness of classification and honesty about what has been proved. Three continuations suggest themselves: the exact confirmation of the 22 bounded entries and the systematic comparison of all tables against the best-known-codes records; the extension of the programme to GL(4,2), where the exceptional isomorphism with the alternating group of degree 8 links the present constructions to the alternating-group literature; and the companion study of the combinatorial designs supported by the same primitive actions through the Key-Moori methods.

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