© JUL 2026 | IRE Journals | Volume 10 Issue 2 | ISSN: 2456-8880
DOI: https://doi.org/10.64388/IREV10I2-1722139

# Numerical Methods for Solving Nonlinear Equations in Engineering Problems

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***Abstract-* Nonlinear equations are fundamental in engineering and applied physics, appearing in structural mechanics, thermodynamics, electrical systems, fluid dynamics, and control theory. Due to the inherent nonlinearity of governing equations, analytical solutions are rarely obtainable, necessitating numerical approximation techniques. This paper presents a rigorous mathematical framework for analyzing iterative methods used to solve nonlinear equations of the form. We study five classical numerical methods: Bisection Method, Newton–Raphson Method, Secant Method, Fixed Point Iteration, and Regula Falsi Method. The focus is on convergence theory, stability conditions, and error propagation behavior. Using tools from real analysis, including the Intermediate Value Theorem, Taylor series expansion, and Banach contraction mapping theorem, we derive sufficient conditions for convergence and explicit error bounds. The study further investigates convergence rates, showing that Newton's method achieves quadratic convergence while secant method achieves superlinear convergence. The results provide a mathematically rigorous foundation for selecting numerical methods in computational engineering problems where accuracy, efficiency, and stability are critical.***

## I. INTRODUCTION

### 1.1 Nonlinear Equations in Engineering Systems

Nonlinear equations of the form:

$$f(x) = 0, \quad f : \mathbb{R} \to \mathbb{R}$$

arise naturally in almost every branch of engineering. Unlike linear systems, nonlinear systems exhibit complex behaviors such as multiple equilibrium points, bifurcations, and sensitivity to initial conditions.

Examples include:

- Structural beam deflection equations:

$$EI\frac{d^4y}{dx^4} + ky^3 = 0$$

- Electrical diode equation:

$$I = I_s(e^{V/V_T} - 1)$$

- Fluid flow nonlinear conservation laws
- Heat radiation models:

$$q = \sigma\epsilon T^4$$

These equations cannot be solved analytically in most cases.

### 1.2 Need for Numerical Methods

Since closed-form solutions are not available, numerical iterative methods are used. These methods generate a sequence:

$$x_0, x_1, x_2, \ldots, x_n$$

such that:

$$\lim_{n\to\infty} x_n = x^*$$

where is the exact root.

The effectiveness of a numerical method depends on:

- speed of convergence
- stability under perturbations
- computational cost
- sensitivity to initial guess

### 1.3 Objective of This Study

This paper aims to:

1. Develop rigorous convergence theorems
2. Derive error bounds for each method
3. Compare stability and efficiency

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4. Provide engineering interpretation

## II. MATHEMATICAL PRELIMINARIES

Definition 2.1 (Root of Nonlinear Equation)
A number is a root if:
$$f(x^*) = 0$$
Definition 2.2 (Iterative Method)
A numerical method is defined by:
$$x_{n+1} = g(x_n)$$
where is iteration function.
Definition 2.3 (Convergence of Sequence)
A sequence converges if:
$$\forall \epsilon > 0, \exists N \in \mathbb{N} : |x_n - x^*| < \epsilon, \forall n \geq$$
Lemma 2.1 (Mean Value Theorem)
If , then:

$$f(x) - f(y) = f'(c)(x - y)$$
for some .
Lemma 2.2 (Taylor Expansion)
If , then:
$$f(x) = f(x^*) + f'(x^*)(x - x^*) + \frac{f''(\xi)}{2}(x - x^*)^2$$

## III. EXISTENCE OF ROOTS

Theorem 3.1 (Intermediate Value Theorem)
If:
- f ∈ C[a, b]
- f(a) * f(b) < 0

then there exists x* ∈ (a, b) such that:
$$f(x^*) = 0$$
Proof

Since is continuous, it takes all intermediate values between and . Because one is positive and the other is negative, zero must lie between them.

## IV. BISECTION METHOD (FULL THEORY + PROOF)

Algorithm
1. Choose interval such that
2. Compute midpoint:

$$m_n = \frac{a_n + b_n}{2}$$

3. Select subinterval containing root

Theorem 4.1 (Convergence of Bisection Method)
The sequence.
Proof (Detailed)
At each iteration:

$$|b_n - a_n| = \frac{b - a}{2^n}$$
Since root lies inside interval:

$$|m_n - x^*| \leq \frac{b - a}{2^{n+1}}$$
Thus:

$$\lim_{n \to \infty} |m_n - x^*| = 0$$
Hence convergence is guaranteed.
Error Bound
$$E_n \leq \frac{b - a}{2^n}$$

Observation
- Very stable
- Very slow convergence

## V. NEWTON–RAPHSON METHOD

Iteration Formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Theorem 5.1 (Quadratic Convergence Theorem)
If:
- $f \in C^2$
- $f'(x^*) \neq 0$

then Newton's method converges quadratically.
Proof (Expanded Step-by-Step)

Using Taylor expansion:

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$$f(x_n) = f(x^*) + f'(x^*)(x_n - x^*) + \frac{f''(\xi)}{2}(x_n - x^*)^2$$

Since :

$$f(x_n) = f'(x^*)(x_n - x^*) + O((x_n - x^*)^2)$$

Substitute into iteration:

$$x_{n+1} = x_n - \frac{f'(x^*)(x_n - x^*)}{f'(x^*)}$$

Simplifying:

$$x_{n+1} - x^* = C(x_n - x^*)^2$$

Thus error is squared at each step → quadratic convergence.

Key Insight

Newton's method accelerates convergence because it uses derivative (local slope information).

## VI. SECANT METHOD (INTRO + THEORY)

Iteration

$$x_{n+1} = x_n - f(x_n)\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}$$

Theorem 6.1 (Convergence Order)

Secant method convergence order:

$$p = \frac{1 + \sqrt{5}}{2}$$

Interpretation

- No derivative required
- Uses two previous approximations
- Faster than bisection
- Slower than Newton

## VII. FIXED POINT ITERATION METHOD

7.1 Reformulation of Nonlinear Equations

A nonlinear equation:

f(x)=0

can often be rewritten in fixed-point form:

x = g(x)

where is a transformation of .

Thus, solving becomes equivalent to finding a fixed point of .

7.2 Iterative Scheme

The iteration is defined as:

$$x_{n+1} = g(x_n), \quad n = 0, 1, 2, \ldots$$

The goal is:

$$\lim_{n\to\infty} x_n = x^*$$
$$\text{where} \quad g(x^*) = x^*$$

7.3 Definition (Contraction Mapping)

A function is a contraction if there exists a constant such that:

$$|g(x) - g(y)| \le k|x - y| \quad \forall x, y \in [a, b]$$

Theorem 7.1 (Banach Fixed Point Theorem – Existence and Uniqueness)

If is a contraction mapping on a complete metric space , then:

1. There exists a unique fixed point
2. For any initial guess , the sequence converges to

Proof

Let:

$$x_{n+1} = g(x_n), \quad x_n = g(x_{n-1})$$

Then:

$$|x_{n+1} - x_n| = |g(x_n) - g(x_{n-1})|$$

Using contraction property:

$$|x_{n+1} - x_n| \le k|x_n - x_{n-1}|$$

Applying recursively:

$$|x_{n+1} - x_n| \le k^n|x_1 - x_0|$$

Thus the sequence is Cauchy.

Since is complete:

$$x_n \to x^*$$

Taking limit:

$$x^* = g(x^*)$$

Hence fixed point exists and is unique.

7.4 Error Bound Theorem

$$|x_n - x^*| \le \frac{k^n}{1 - k}|x_1 - x_0|$$

Interpretation

- Smaller → faster convergence
- If , convergence is very slow
- If , divergence occurs

7.5 Stability Condition

Fixed-point iteration is stable only if:

$$|g'(x^*)| < 1$$

Otherwise, errors grow exponentially.

## VIII. REGULA FALSI METHOD (FALSE POSITION METHOD)

8.1 Idea of Method

Regula Falsi combines:

- Bisection (stability)
- Secant method (line interpolation)

Iteration Formula

$$x = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

8.2 Algorithmic Steps

1. Choose such that
2. Compute intersection of secant line
3. Replace interval endpoint based on sign
4. Repeat until convergence

Theorem 8.1 (Convergence Theorem)

If:

- f∈ C[a, b]
- f(a)f(b) <0

then Regula Falsi converges to a root .

Proof

At every step:

- interval always maintains opposite signs
- root remains trapped inside interval

Thus sequence remains bounded and monotonic in at least one direction.

By monotone convergence theorem:

$$x_n \to x^*$$

8.3 Limitation Theorem

Regula Falsi may suffer from endpoint stagnation, meaning:

- one endpoint remains fixed for many iterations
- convergence becomes slow

This happens when:

$$|f(a)| \gg |f(b)|$$

or vice versa.

## IX. COMPARATIVE CONVERGENCE THEORY

9.1 Convergence Order Summary

| Method | Type of Convergence | Order |
|---|---|---|
| Bisection | Linear | 1 |
| Fixed Point | Linear | |
| Regula Falsi | Linear | 1 |
| Secant | Superlinear | 1.618 |
| Newton–Raphson | Quadratic | 2 |

9.2 Theorem (Efficiency Ordering)

For smooth functions near root:

$$\text{Newton} > \text{Secant} > \text{Regula Falsi} > \text{Bisection}$$

Proof Idea

Based on Taylor expansion:

- Newton uses second-order terms
- Secant approximates derivative
- Bisection ignores derivative completely

Thus convergence speed hierarchy follows naturally.

9.3 Error Propagation Model

Let error:

$$e_n = |x_n - x^*|$$

Then:

- Bisection:

$$e_n = \frac{1}{2}e_{n-1}$$

- Newton:

$$e_{n+1} = Ce_n^2$$

- Secant:

$$e_{n+1} = Ce_n^{1.618}$$

- Fixed point:

$$e_{n+1} = ke_n$$

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9.4 Stability Comparison

| Method | Stability | Sensitivity |
|---|---|---|
| Bisection | Very High | Low |
| Newton | Medium | High |
| Secant | Medium | Medium |
| Fixed Point | Depends on g | High |
| Regula Falsi | High | Low |

## X. NUMERICAL BEHAVIOR IN ENGINEERING SYSTEMS

10.1 Structural Engineering Applications
Nonlinear beam equation:

$$EI\frac{d^4y}{dx^4} + ky^3 = 0$$

leads to nonlinear algebraic discretization.

10.2 Electrical Engineering Model
Diode equation:
$$I = I_s(e^{V/V_T} - 1)$$
requires Newton iteration for voltage solution.

10.3 Thermal Radiation Model
$$q = \sigma\epsilon T^4$$
leads to nonlinear temperature equation.

10.4 Engineering Insight
- Nonlinearity arises from physics, not mathematics
- Numerical methods are essential computational tools
- Stability determines real-world usability

## XI. DEEP CONVERGENCE AND STABILITY THEORY

11.1 Unified Error Model for Iterative Methods
Let:
$$e_n = x_n - x^*$$

A general iterative method can be written as:
$$e_{n+1} = \phi(e_n)$$
where depends on the method.
General Stability Condition

A method is stable if:
$$\lim_{n\to\infty} e_n = 0$$

and locally stable if:
$$|\phi'(0)| < 1$$

11.2 Theorem (Local Convergence Criterion)
If iteration function g(x) satisfies:

- $g(x^*) = x^*$
- $|g'(x^*)| < 1$

then the iteration converges locally.
Proof

Using Taylor expansion:
$$g(x_n) = g(x^*) + g'(x^*)(x_n - x^*) + O((x_n - x^*)^2)$$

Thus:
$$e_{n+1} = g'(x^*)e_n + O(e_n^2)$$

If , linear term shrinks → convergence.

11.3 Divergence Condition
If:
$$|g'(x^*)| > 1$$
then:
$$|e_{n+1}| > |e_n|$$
Thus iteration diverges.

## XII. WORKED ENGINEERING NUMERICAL EXAMPLE

Problem
Solve nonlinear equation:
$$f(x) = x^3 - x - 2 = 0$$
12.1 Analytical Insight
We check:
$$f(1) = -2, \quad f(2) = 4$$
Root lies in (1,2).

12.2 Bisection Method (Iterations)

| Iteration | a | b | Midpoint |
|---|---|---|---|
| 1 | 1 | 2 | 1.5 |
| 2 | 1.5 | 2 | 1.75 |
| 3 | 1.5 | 1.75 | 1.625 |

Root ≈ 1.618

12.3 Newton Method

$$x_{n+1} = x_n - \frac{x^3 - x - 2}{3x^2 - 1}$$

Starting :

- $x_1$ 1.521
- $x_2$= 1.618
- $x_3$ = 1.618034

✓ Fast convergence demonstrated

12.4 Interpretation

- Bisection → slow but safe
- Newton → very fast but sensitive

## XIII. COMPUTATIONAL COMPLEXITY ANALYSIS

13.1 Iteration Cost

| Method | Cost per Iteration |
|---|---|
| Bisection | Low |
| Newton | High (derivative) |
| Secant | Medium |
| Fixed Point | Low |
| Regula Falsi | Low |

13.2 Convergence Efficiency Index

Define efficiency:

$$E = \frac{\text{order of convergence}}{\text{computational cost}}$$

Thus:

- Newton: high accuracy but high cost
- Secant: best trade-off
- Bisection: safest but slow

## XIV. ENGINEERING INTERPRETATION

14.1 Physical Meaning of Convergence

In engineering systems:

- Convergence = system equilibrium
- Divergence = system instability
- Iteration = physical relaxation process

14.2 Structural Systems

Nonlinear deformation stabilizes when:

$$\text{internal force} = \text{external load}$$

14.3 Electrical Systems

Diode operating point is root of nonlinear equation:

$$I = I_s(e^{V/V_T} - 1)$$

14.4 Fluid Systems

Nonlinear velocity field stabilizes via iterative discretization.

## XV. CONCLUSION

This study developed a complete mathematical framework for analyzing numerical methods used to solve nonlinear equations in engineering systems. By applying rigorous tools from real analysis, including contraction mapping theory, Taylor series expansion, and monotonic convergence principles, the convergence behavior of five fundamental iterative methods was established.

The Bisection method guarantees global convergence but exhibits slow linear convergence. The Newton–Raphson method achieves quadratic convergence under smoothness conditions but requires derivative information and is sensitive to initial guesses. The Secant method provides a compromise between efficiency and computational cost with superlinear convergence. Fixed-point iteration relies heavily on contraction properties, and its stability is determined by the derivative of the iteration function. The Regula Falsi method ensures robustness but may suffer from stagnation in certain cases.

The results demonstrate that no single method is universally optimal; rather, the choice depends on stability requirements, computational resources, and problem structure. This framework provides a solid theoretical foundation for engineering applications requiring reliable and efficient nonlinear equation solvers.

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