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Vague Vector Spaces, Vague Normed Spaces, Vague Banach Spaces and Vague Hilbert Spaces: A Unified Mathematical Framework

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Abstract- Classical vector spaces, normed spaces, Banach spaces and Hilbert spaces assume that algebraic membership and metric information are precisely specified. In applications involving incomplete measurements, uncertain coefficients, noisy data and approximate computational states, such precision may not be available. This paper develops a coherent interval-valued vague framework for these structures. A vague value is represented by a truth degree and a falsity degree, while a vague magnitude is represented by a closed interval of nonnegative real numbers. Vague vector spaces are introduced by attaching compatible truth and falsity functions to an ordinary vector space. Vague norms are then defined through lower and upper magnitude functions satisfying positivity, definiteness, absolute homogeneity and componentwise triangle inequalities. Vague convergence, vague Cauchy sequences and vague completeness are formulated using the upper norm. A vague Banach space is defined as a complete vague normed vector space. A corresponding vague inner-product framework is developed, leading to the definition of vague Hilbert spaces. A principal construction is obtained from any classical normed space by the uncertainty envelope ||x||_v=[(1-epsilon)||x||,(1+epsilon)||x||], 0<=epsilon<1. It is proved that this construction preserves convergence, Cauchy sequences and completeness. Finite-dimensional spaces, sequence spaces, continuous-function spaces, L2 spaces and matrix spaces are used as examples. The relationship with classical Banach and Hilbert spaces is established, and applications to uncertain approximation, digital mathematics, matrix analysis and quantum-inspired computation are discussed. The paper also identifies limitations of interval-valued inner products and directions for further research.

Keywords: vague set; vague value; vague vector space; vague norm; vague normed space; vague convergence; vague Cauchy sequence; vague Banach space; vague inner product; vague Hilbert space; interval uncertainty.

I. INTRODUCTION

Functional analysis is built upon a sequence of increasingly rich structures. A vector space supplies linear operations; a normed space supplies a notion of magnitude; a Banach space adds completeness; and a Hilbert space adds an inner-product geometry together with completeness. These concepts are central to approximation theory, differential equations, operator theory, numerical analysis, optimization, signal processing and quantum mechanics.

The classical formulation presumes that the quantities involved are exact. If x is a vector, its norm ||x|| is a single nonnegative real number. If x is a member of a specified class, its membership is ordinarily represented by a binary decision. Practical information is often different. Experimental observations may only bound a magnitude, coefficients may be known approximately, and a computational state may have evidence both for and against a proposition. Vague set theory, introduced in the literature as a means of representing evidence for and against membership, provides one route to modeling such information.

The present paper develops a mathematically explicit framework rather than merely replacing symbols in classical definitions. The algebraic operations remain ordinary vector-space operations, while uncertainty is carried by truth/falsity functions and interval-valued magnitudes. This separation is useful: it prevents uncertainty from being confused with the algebraic operations themselves.

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The principal construction in this article starts with a classical normed space and replaces its exact norm by a controlled uncertainty envelope. For a fixed uncertainty parameter epsilon in [0,1), define ||x||_v=[(1-epsilon)||x||,(1+epsilon)||x||]. This simple model has the advantage that all classical convergence and completeness results can be transferred rigorously. More general variable-width vague norms are discussed as an open direction.

The paper is organized as follows. Section 2 reviews vague values and classical functional-analytic terminology. Section 3 introduces vague vector spaces and subspaces. Section 4 develops vague norms and vague normed spaces. Section 5 treats convergence, Cauchy sequences and completeness. Section 6 constructs vague Banach spaces. Section 7 introduces vague inner products and vague Hilbert spaces. Sections 8–10 give examples and structural results. Section 11 discusses applications and Section 12 gives limitations and future work.

II. PRELIMINARIES

2.1 Classical vector spaces
Let F be R or C. A vector space V over F is a nonempty set equipped with addition and scalar multiplication satisfying the usual axioms. The zero vector is denoted by 0. If x belongs to V, then -x denotes its additive inverse.

2.2 Normed spaces
A norm on V is a mapping ||.||:V -> [0,infinity) satisfying, for all x,y in V and alpha in F, (i) ||x||>=0; (ii) ||x||=0 if and only if x=0; (iii) ||alpha x||=|alpha| ||x||; and (iv) ||x+y|| <= ||x||+||y||. A normed vector space is the pair (V,||.||).

2.3 Banach spaces
A sequence {x_n} in a normed space is Cauchy when, for every epsilon>0, there exists N such that ||x_m-x_n||<epsilon for all m,n>=N. A Banach space is a normed space in which every Cauchy sequence converges to a member of the space.

2.4 Hilbert spaces
An inner product on a real vector space is a symmetric, bilinear, positive-definite form. On a complex vector space it is sesquilinear and conjugate symmetric. It induces the norm ||x||=sqrt(<x,x>). A complete inner-product space is a Hilbert space.

2.5 Vague values
Let X be a nonempty universe. For x in X, let t(x), f(x) belong to [0,1] and satisfy t(x)+f(x)<=1. The value t(x) is interpreted as evidence supporting membership and f(x) as evidence supporting non-membership. The associated interval is V(x)=[t(x),1-f(x)]. Its width is w(x)=1-f(x)-t(x).

V(x) = [t(x), 1 - f(x)].

The width w(x) measures unresolved information. Width zero corresponds to an exact membership value. This representation is closely related to the classical vague-set viewpoint, but the functional-analytic constructions below are stated independently so that their mathematical assumptions are transparent.

III. VAGUE VECTOR SPACES

3.1 Definition
Let V be a vector space over F. A vague vector structure on V consists of functions mu_V, nu_V:V->[0,1] satisfying 0<=mu_V(x)+nu_V(x)<=1 for all x. We interpret mu_V(x) as the degree of positive evidence attached to x and nu_V(x) as the degree of negative evidence.

0 <= mu_V(x)+nu_V(x) <= 1.

For compatibility with the vector operations, impose mu_V(x+y)>=min{mu_V(x),mu_V(y)} and nu_V(x+y)<=max{nu_V(x),nu_V(y)}. For nonzero scalars alpha, require mu_V(alpha x)>=mu_V(x) and nu_V(alpha x)<=nu_V(x). We additionally impose symmetry under additive inversion when required: mu_V(-x)=mu_V(x), nu_V(-x)=nu_V(x).

Definition 3.1
A vague vector space is a vector space V over F together with functions mu_V and nu_V satisfying the preceding consistency conditions.

3.2 Constant vague structures

The simplest examples are obtained by taking constants a,b in [0,1] with a+b<=1 and setting mu_V(x)=a and nu_V(x)=b for every x. All compatibility inequalities then hold as equalities.

Theorem 3.1

Every vector space V equipped with constant functions mu_V(x)=a and nu_V(x)=b, where a+b<=1, is a vague vector space.

Proof. For x,y in V, mu_V(x+y)=a=min{a,a} and nu_V(x+y)=b=max{b,b}. For every nonzero alpha, mu_V(alpha x)=a=mu_V(x) and nu_V(alpha x)=b=nu_V(x). Hence all conditions are satisfied.

3.3 Zero vector

Theorem 3.2

If a vague vector structure satisfies mu_V(-x)=mu_V(x) and nu_V(-x)=nu_V(x), then mu_V(0)>=mu_V(x) and nu_V(0)<=nu_V(x) for every x.

Proof. Since 0=x+(-x), the addition inequalities give mu_V(0)>=min{mu_V(x),mu_V(-x)}=mu_V(x), and nu_V(0)<=max{nu_V(x),nu_V(-x)}=nu_V(x).

3.4 Vague subspaces

A nonempty subset W of V is called a vague subspace when W is a classical vector subspace and the restrictions of mu_V and nu_V to W satisfy the vague vector conditions. Thus a vague subspace inherits both algebraic and uncertainty information.

Theorem 3.3

If W is a vector subspace of a vague vector space V, then W with restricted functions is a vague vector space.

Proof. Closure under addition and scalar multiplication follows from W being a vector subspace. The vague inequalities are inherited by restriction. Therefore W satisfies the required axioms.

IV. VAGUE NORMED SPACES

4.1 Interval-alued vague norm

A vague norm should represent uncertain magnitude while retaining the essential geometry of a norm. Let I(R_+) denote the set of closed intervals [a,b] with 0<=a<=b. A vague norm is a map ||.||_v:V->I(R_+), written ||x||_v=[||x||_-,||x||_+].

Definition 4.1

A mapping ||.||_v is a vague norm if for all x,y in V and alpha in F: (VN1) 0<=||x||_-<=||x||_+; (VN2) ||x||_- = ||x||_+ =0 if and only if x=0; (VN3) ||alpha x||_v=|alpha| ||x||_v componentwise; and (VN4) ||x+y||_-<=||x||_-+||y||_- and ||x+y||_+<=||x||_++||y||_+.

The use of componentwise inequalities is deliberate. It gives a sufficient operational form without requiring a general theory of interval addition and ordering beyond the nonnegative real line.

4.2 Fundamental construction

Let (V,||.||) be a classical normed space and choose epsilon in [0,1). Define
||x||_v = [(1-epsilon)||x||, (1+epsilon)||x||].

Theorem 4.1

The preceding formula defines a vague norm on V.

Proof. The lower endpoint is nonnegative and does not exceed the upper endpoint because 1-epsilon<=1+epsilon. If ||x||_v=[0,0], then (1-epsilon)||x||=0. Since epsilon<1, ||x||=0 and x=0. Conversely x=0 gives [0,0]. For homogeneity, ||alpha x||_v=[(1-epsilon)|alpha| ||x||,(1+epsilon)|alpha| ||x||]=|alpha| ||x||_v. Finally, by the classical triangle inequality, ||x+y||<=||x||+||y||; multiplying by each positive factor gives the required lower and upper component estimates.

4.3 Vague normed space

A vector space equipped with a vague norm is called a vague normed space. The underlying linear structure is exact, while the magnitude assigned to each vector is interval-valued.

Example 4.1: R^2

For x=(x1,x2), let ||x||_2=sqrt(x1^2+x2^2). With epsilon=0.1, ||x||_v=[0.9||x||_2,1.1||x||_2]. For x=(3,4),

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the classical norm is 5 and the vague norm is $[4.5, 5.5]$.

4.4 Matrix spaces

For A in $M_{m,n}(R)$, the Frobenius norm is $\|A\|_F=(\sum_i \sum_j a_{ij}^2)^{1/2}$. The induced vague matrix norm is $[(1-\epsilon)\|A\|_F, (1+\epsilon)\|A\|_F]$. This is useful for uncertain matrix computations.

Example 4.2

For the identity matrix I_2, $\|I_2\|_F=\sqrt{2}$. With $\epsilon=0.1$, the vague norm is approximately $[1.273, 1.556]$.

V. VAGUE METRIC, CONVERGENCE AND COMPLETENESS

5.1 Vague metric

The upper component of a vague norm provides a convenient metric.

$$d_v(x,y)=\|x-y\|_+.$$

Theorem 5.1

If $\|.\|_v$ is a vague norm, then $d_v(x,y)=\|x-y\|_+$ is a metric.

Proof. Nonnegativity is immediate. If $d_v(x,y)=0$, then $\|x-y\|_+=0$ and definiteness gives $x=y$. Symmetry follows from $\|x-y\|_+=\|y-x\|_+$. For the triangle inequality, $x-z=(x-y)+(y-z)$, hence $\|x-z\|_+\leq\|x-y\|_++\|y-z\|_+$.

5.2 Vague convergence

A sequence $\{x_n\}$ vaguely converges to x when $\|x_n-x\|_+$ tends to zero. We write $x_n \to_v x$. A stronger interval-collapse interpretation additionally requires the lower endpoint to tend to zero.

$$x_n \to_v x \text{ iff } \lim_{n\to\infty} \|x_n-x\|_+ = 0.$$

Theorem 5.2

Vague limits are unique.

Proof. If $x_n \to_v x$ and $x_n \to_v y$, then $d_v(x,y)\leq d_v(x,x_n)+d_v(x_n,y)$. Taking limits gives $d_v(x,y)=0$, hence $x=y$.

5.3 Vague Cauchy sequences

A sequence is vague Cauchy if for every $\delta>0$ there exists N such that $\|x_m-x_n\|_+<\delta$ whenever $m,n\geq N$.

Theorem 5.3

Every vaguely convergent sequence is vague Cauchy.

Proof. If $x_n \to_v x$, choose N such that $\|x_n-x\|_+<\delta/2$ for $n\geq N$. Then the triangle inequality gives $\|x_m-x_n\|_+<\delta$.

5.4 Vague completeness

A vague normed space is vaguely complete if every vague Cauchy sequence converges vaguely to an element of the same space. A vague Banach space is a vaguely complete vague normed vector space.

VI. VAGUE BANACH SPACES

6.1 Transfer theorem

Theorem 6.1

Let $(V,\|.\|)$ be a Banach space. For fixed ϵ in $[0,1)$, equip V with $\|x\|_v=[(1-\epsilon)\|x\|,(1+\epsilon)\|x\|]$. Then V is a vague Banach space.

Proof. Suppose $\{x_n\}$ is vague Cauchy. For any $\delta>0$, eventually $(1+\epsilon)\|x_m-x_n\|<\delta$. Hence $\|x_m-x_n\|<\delta/(1+\epsilon)$, so $\{x_n\}$ is classically Cauchy. Completeness of V gives x in V with $\|x_n-x\|\to0$. Therefore $\|x_n-x\|_+=(1+\epsilon)\|x_n-x\|\to0$, so $x_n \to_v x$.

6.2 Converse transfer

Theorem 6.2

Under the same fixed-epsilon construction, V is a classical Banach space if and only if its induced vague normed space is vague Banach.

Proof. The forward implication is Theorem 6.1. Conversely, every classical Cauchy sequence is vague Cauchy because $\|x_m-x_n\|_+=(1+\epsilon)\|x_m-x_n\|$. Vague completeness therefore gives a vague limit, which is also a classical limit because the upper norm is a positive constant multiple of the classical norm.

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6.3 Finite-dimensional spaces

Every finite-dimensional normed vector space over R or C is complete. Consequently, R^n and C^n with any fixed induced vague norm are vague Banach spaces.

6.4 Sequence spaces

For $1<=p<\infty$, the classical space ℓ^p is Banach. Its induced vague norm is $\|x\|_{v,p}=[(1-\epsilon)\|x\|_p,(1+\epsilon)\|x\|_p]$. The space ℓ^∞ is treated similarly.

Example 6.1

For ℓ^1, $\|x\|_1=\sum_n |x_n|$ and $\|x\|_{v,1}=[(1-\epsilon)\|x\|_1,(1+\epsilon)\|x\|_1]$. This is a vague Banach space but, in general, not a vague Hilbert space with this norm.

6.5 Continuous functions

The classical Banach space C[a,b] with supremum norm $\|f\|_\infty=\max_{a<=x<=b}|f(x)|$ yields the vague norm $\|f\|_{v,\infty}=[(1-\epsilon)\|f\|_\infty,(1+\epsilon)\|f\|_\infty]$.

VII. VAGUE INNER-PRODUCT SPACES AND HILBERT SPACES

7.1 Motivation

Norms measure magnitude but do not by themselves encode angles or orthogonality. An inner product supplies these geometric notions. To avoid ambiguities in interval-valued complex arithmetic, the core definition in this paper is first formulated for real inner-product spaces.

Definition 7.1

A vague inner-product structure on a real vector space assigns an interval to each pair (x,y), written $\langle x,y\rangle_v=[L(x,y),U(x,y)]$, together with rules guaranteeing symmetry, compatible linearity, and positive definiteness in the exact zero-uncertainty case. For the principal induced construction below, the interval is generated from a classical inner product.

7.2 Induced vague inner product

Let H be a real inner-product space with classical inner product $\langle x,y\rangle$. For ϵ in [0,1), define

$$\langle x,y\rangle_v = [\langle x,y\rangle - \epsilon |\langle x,y\rangle|, \langle x,y\rangle + \epsilon |\langle x,y\rangle|].$$

This definition is sign-safe: if the exact inner product is negative, the interval remains ordered. When $\langle x,x\rangle=\|x\|^2$, the induced squared magnitude lies in $[(1-\epsilon)\|x\|^2,(1+\epsilon)\|x\|^2]$.

7.3 Vague Hilbert space

A vague Hilbert space is a vague inner-product space that is complete with respect to the metric induced by its vague norm. In the induced construction, the norm is generated from the interval of $\langle x,x\rangle_v$ or, equivalently, by a fixed uncertainty envelope around the classical Hilbert norm.

Theorem 7.1

Every classical real Hilbert space admits the fixed-epsilon induced vague Hilbert structure.

Proof. The classical inner product supplies the exact geometry, while the induced vague norm is a constant-factor equivalent of the classical norm. Since a Hilbert space is complete, Theorem 6.1 gives vague completeness. Hence the resulting structure is a vague Hilbert space.

7.4 Euclidean vague Hilbert spaces

For R^n, $\langle x,y\rangle=\sum_i x_i y_i$ and $\|x\|_2=\sqrt{\langle x,x\rangle}$. The induced vague norm is $[(1-\epsilon)\|x\|_2,(1+\epsilon)\|x\|_2]$. Thus R^n is a finite-dimensional vague Hilbert space.

7.5 Vague orthogonality

In the strong exact case, x and y are vaguely orthogonal when $\langle x,y\rangle_v=[0,0]$. This occurs whenever the underlying exact inner product is zero. A more permissive interval notion, in which zero belongs to the interval, can model uncertain orthogonality but requires additional choices about how interval signs are interpreted.

VIII. INFINITE-DIMENSIONAL VAGUE HILBERT SPACES

8.1 The space ℓ^2

The sequence space ℓ^2 consists of all $x=(x_n)$ for which $\sum_n |x_n|^2<\infty$. Its classical inner

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product is <x,y>=sum_n x_n y_n in the real case. It is a Hilbert space.

||x||_2 = (sum_{n=1}^infinity |x_n|^2)^(1/2).
The vague norm ||x||_{v,2}=[(1-epsilon)||x||_2,(1+epsilon)||x||_2] therefore yields an infinite-dimensional vague Hilbert space.

8.2 Example
Let x=(1,1/2,1/3,...). Since sum_n 1/n^2 converges, x belongs to ell^2. Its vague magnitude is the interval obtained by multiplying the classical square-summable norm by 1-epsilon and 1+epsilon.

8.3 The space L^2[a,b]
The classical space L^2[a,b] has inner product <f,g>=integral_a^b f(x)g(x)dx and norm ||f||_2=(integral_a^b |f(x)|^2dx)^(1/2). Its induced vague norm gives a vague Hilbert space.

8.4 Orthonormal families
If {e_n} is an orthonormal family in a Hilbert space, then <e_m,e_n>=delta_mn. Under the induced vague inner product, exact orthogonality remains represented by [0,0] for m not equal to n, while unit length is represented by an interval around 1 for m=n.

IX. STRUCTURAL RESULTS

9.1 Equivalence of convergence
Theorem 9.1
For the fixed-epsilon vague norm, x_n ->_v x if and only if x_n -> x classically.

Proof. The upper vague norm is (1+epsilon)||x_n-x||. Since 1+epsilon is a fixed positive constant, one expression tends to zero exactly when the other does.

9.2 Equivalence of Cauchy properties
Theorem 9.2
A sequence is vague Cauchy under the fixed-epsilon norm if and only if it is classically Cauchy.

Proof. The condition ||x_m-x_n||_+<delta is equivalent to ||x_m-x_n||<delta/(1+epsilon).

9.3 Continuity of bounded operators
Let T:V->W be linear. If there is C>0 such that ||Tx||_{W,+}<=C||x||_{V,+}, then T is vaguely continuous. Indeed, ||Tx-Ty||_{W,+}<=C||x-y||_{V,+}.

9.4 Vague contraction principle
Suppose T is a self-map of a vague Banach space satisfying ||Tx-Ty||_+<=k||x-y||_+ for 0<=k<1. The standard iterative proof gives a unique fixed point, because the iterates form a vague Cauchy sequence and the space is vaguely complete.

Theorem 9.3
A contraction on a vague Banach space, with respect to the upper vague metric, has a unique fixed point.
Proof. Starting with x_0, define x_{n+1}=T(x_n). Then ||x_{n+1}-x_n||_+<=k^n||x_1-x_0||_+. Summing the geometric series shows that {x_n} is vague Cauchy. Completeness gives x*. Continuity of T gives T(x*)=x*. Uniqueness follows from ||x*-y*||_+<=k||x*-y*||_+.

X. EXAMPLES AND COMPARISON

<table>
<tr><th>Space</th><th>Classical norm</th><th>Vague norm</th><th>Structure</th></tr>
<tr><td>R^n</td><td>Euclidean norm</td><td>[(1-eps)||x||_2,(1+eps)||x||_2]</td><td>Vague Banach + vague Hilbert</td></tr>
<tr><td>ell^1</td><td>sum |x_n|</td><td>[(1-eps)||x||_1,(1+eps)||x||_1]</td><td>Vague Banach</td></tr>
<tr><td>ell^2</td><td>(sum |x_n|^2)^1/2</td><td>[(1-eps)||x||_2,(1+eps)||x||_2]</td><td>Vague Hilbert</td></tr>
<tr><td>C[a,b]</td><td>sup |f(x)|</td><td>[(1-eps)||f||_infty,(1+eps)||f||_infty]</td><td>Vague Banach</td></tr>
<tr><td>L^2[a,b]</td><td>(integral |f|^2)^1/2</td><td>[(1-eps)||f||_2,(1+eps)||f||_2]</td><td>Vague Hilbert</td></tr>
<tr><td>M_m,n(R)</td><td>Frobenius norm</td><td>[(1-eps)||A||_F,(1+eps)||A||_F]</td><td>Vague Banach + Hilbert</td></tr>
</table>

Table 1. Representative spaces in the fixed-width vague framework.

10.1 Classical reduction
When epsilon=0, the interval collapses: ||x||_v=[||x||,||x||]. Therefore the vague theory reduces exactly to the classical normed-space theory. This reduction is an important consistency criterion.

Theorem 10.1
The classical Banach and Hilbert structures are special cases of the proposed vague framework.

Proof. Set epsilon=0. All vague norms become singleton intervals and vague convergence becomes ordinary norm convergence. The corresponding completeness and inner-product structures therefore coincide with the classical definitions.

10.2 Banach versus Hilbert
Every vague Hilbert space constructed from a Hilbert space is also a vague Banach space because its inner-product norm is a norm and completeness is retained. The converse need not hold. The classical example ell^1 is Banach but its usual norm is not induced by an inner product.

10.3 Polynomial spaces
The finite-dimensional polynomial space P_n can be identified with R^{n+1} through coefficient vectors. A vague coefficient norm is therefore obtained immediately from the Euclidean norm of the coefficient vector. This provides a simple model for uncertain polynomial approximation.

10.4 Graph and matrix spaces
Finite graph-induced vector spaces and matrix spaces can be equipped with the same vague norm construction. If a graph subspace is represented by vectors in R^m, then uncertainty in graph weights or measurements can be represented by an interval-valued norm. Matrix spaces similarly support vague Frobenius geometry.

XI. APPLICATIONS

11.1 Uncertain numerical computation
Numerical data often have tolerances. If an exact computed vector has norm r but the measurement or numerical uncertainty is bounded by epsilon, the interval [(1-epsilon)r,(1+epsilon)r] provides a transparent magnitude envelope.

11.2 Approximation theory
Given a subspace M of a vague normed space, an uncertain approximation problem can be written as minimizing the upper distance ||x-y||_+ over y in M. This formulation separates guaranteed or conservative error from nominal error.

11.3 Digital mathematics
Digital systems use discrete states such as 0 and 1. In uncertain digital computation, a state can be accompanied by truth and falsity evidence. Vague vector structures over finite state spaces could be investigated for uncertain Boolean computation, finite automata and error-tolerant logic.

11.4 Matrix analysis
Uncertain matrices occur in numerical linear algebra, control and data processing. The vague Frobenius norm supplies an interval description of matrix magnitude while retaining linear matrix operations.

11.5 Signal processing
Function spaces such as L^2 and sequence spaces are fundamental to signal analysis. A vague norm can represent uncertainty in energy or amplitude measurements. Further research is needed for rigorous vague Fourier, wavelet and operator theories.

11.6 Quantum-inspired mathematics
Hilbert spaces are central to quantum mechanics. The present framework should be viewed as a mathematical model for uncertain or approximate Hilbert-space information rather than as a replacement for standard quantum theory. Vague vectors and matrix transformations may be useful in studying computational models in which amplitudes or measurements are uncertain.

11.7 Optimization and fixed points
The vague contraction principle and vague norm minimization suggest extensions of classical optimization and fixed-point methods. Such extensions may be relevant when objective values,

constraints or distances are known only within bounds.

XII. LIMITATIONS AND FURTHER RESEARCH

The terminology 'vague vector space', 'vague normed space', 'vague Banach space' and 'vague Hilbert space' is not a single universally standardized theory. Existing literature contains related constructions under vague sets, intuitionistic fuzzy sets, interval-valued fuzzy sets and other uncertainty frameworks. Consequently, the definitions in this paper should be regarded as a coherent proposed framework and should be compared systematically with prior definitions before a research claim of novelty is made.

A second limitation concerns interval-valued inner products, especially over complex scalars. The real case is mathematically cleaner. A fully satisfactory complex theory requires a carefully defined interval arithmetic compatible with conjugation, scalar multiplication and positivity.

A third limitation is the use of a fixed uncertainty parameter epsilon. More flexible models should permit vector-dependent or state-dependent uncertainty widths, for example ||x||_v=[(1-epsilon(x))||x||,(1+epsilon(x))||x||], subject to conditions guaranteeing definiteness and the triangle inequality.

Further work should include: (i) intrinsic axiomatizations independent of an underlying classical norm; (ii) variable-width vague norms; (iii) vague dual spaces and weak convergence; (iv) vague compactness and reflexivity; (v) vague operator and spectral theory; (vi) vague Sobolev spaces; (vii) vague differential equations; (viii) vague tensor and matrix analysis; and (ix) rigorous applications to uncertain digital and quantum-inspired systems.

XIII. CONCLUSIONS

A unified framework for vague vector spaces, vague normed spaces, vague Banach spaces and vague Hilbert spaces has been developed. The construction separates exact algebraic operations from uncertain membership and magnitude information. Vague norms are represented by intervals, and convergence and completeness are defined using the upper magnitude component.

The fixed-width construction ||x||_v=[(1-epsilon)||x||,(1+epsilon)||x||] is particularly useful because it transfers the standard theory directly. A Banach space becomes a vague Banach space, and a Hilbert space becomes a vague Hilbert space. Vague convergence and vague Cauchy properties are equivalent to their classical counterparts under this construction, and the classical theory is recovered at epsilon=0.

Examples from finite-dimensional Euclidean spaces, sequence spaces, continuous-function spaces, L^2 spaces and matrix spaces demonstrate the breadth of the framework. The theory also suggests applications in uncertain approximation, numerical computation, digital mathematics, matrix analysis, signal processing and quantum-inspired mathematics.

The main mathematical opportunity is now to move beyond the fixed-width construction and establish an intrinsic vague functional analysis in which uncertainty itself becomes part of the axiomatic structure. Such a theory could provide a useful bridge between classical functional analysis and modern mathematical models of incomplete or uncertain information.

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