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An Optimized Two-Step Hybrid Block Method with One Optimized and Three Free Parameter Hybrid Points for the Numerical Solution of Volterra Integral Equations of the Second Kind

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Abstract- An optimized two-step hybrid block approach with four hybrid points—consisting of one optimized hybrid point and three active free parameters for the numerical solution of second-kind Volterra integral equations is presented in this study. The suggested approach is developed by combining an appropriate power series with exponential fitting as the basis function to construct an effective numerical discrete scheme of the Volterra type that can approximate the solution of the Volterra integral equation of the second kind. While the unknown coefficients and the optimized hybrid point are found by applying suitable consistency requirements and equating the principal term of the local truncation error to zero, the remaining three off-grid points are integrated into the system as flexible free parameters. The continuous formulation of the approach is obtained using a collocation and interpolation procedure. The accuracy and computational efficiency of the final scheme are significantly enhanced by isolating and optimizing the key target off-grid point. Through rigorous Taylor series expansion of the three-step framework, the optimized value of this specific parameter is determined to be 47. The theoretical 400properties of the method, including consistency, convergence, zero-stability, gion of absolute stability, are established. Numerical experiments involving several test problems are conducted to assess the performance of the proposed method in comparison with existing numerical methods. The computed results, absolute errors, and error norms demonstrate that the proposed optimized three-step hybrid block method provides improved accuracy and competitive computational efficiency. The method is therefore shown to be a reliable and effective computational technique for solving Volterra integral equations of the second kind, particularly for problems requiring high accuracy and enhanced stability.

Keywords: Volterra integral equations; second kind; two-step method; optimized hybrid point; free parameters; exponential fitting.

I. INTRODUCTION

Fourth-order ordinary differential equations (ODEs) represent a highly significant family of higher-order equations widely applied across physics, engineering, fluid dynamics, structural mechanics, control systems, and vibration analysis. Because of their greater derivatives, nonlinearities, stiffness, and associated computing expenses, their numerical treatment is frequently difficult. Historically, it was widely believed that higher-order differential equations could only be solved numerically by reducing them to a first-order system of equations. However, as noted by various researchers, this classical reduction technique introduces serious shortcomings, including a complete loss of physical interpretation of the parameters, a massive increase in the dimension of the algebraic system, and rapid error propagation.

To overcome these structural limitations, researchers began examining direct methods for solving higher-order ordinary differential equations. Among these direct approaches, the application of linear multistep predictor-corrector methods gained early attention. While predictor-corrector schemes offered improved stability for stiff systems and flexible combinations of explicit and implicit solvers, they suffered from significant setbacks. They demand intensive human effort during formulation, impose a heavy computational burden due to the repetitive evaluation of predictors per step, and require highly complicated computer programming that lengthens processing execution times.

To completely bypass the computational bottlenecks of predictor-corrector configurations, modern numerical analysts have embraced predictor-free block and hybrid block methods. Block approaches represent an efficient paradigm where solutions at multiple time points are calculated simultaneously within a single computational step. Scholars have successfully advanced block method development for direct solutions of ordinary differential equations by utilizing distinct structural techniques. For instance,

[24] introduced a hybrid numerical technique with block extensions for direct third-order ODE solutions using approximate power series as interpolation equations alongside derivatives as collocation equations. Similarly, [26] utilized Chebyshev series as fundamental base functions to generate interpolation and collocation equations in continuous hybrid linear multistep method development.

Several other authors have contributed hybrid and block formulations for related problems. Akinnukawe and Okunuga [1] proposed a one-step block scheme with an optimal hybrid point for second-order ODEs, while [6] developed a maximal order block method for the same for the same class of problems. Obarhua [10] extended this line of work with a three-step four-point optimized hybrid block method for third-order equations, and [8] presented a four-point block method for direct third-order solutions. Raymond and Kyagya [15] also proposed a three-step two-hybrid block method for second-order ODEs. On the stability and adaptivity side, [16, 17] and [17, 19] introduced optimized and adaptive step-size hybrid block methods with strong stability properties for stiff differential systems. Dibal and Yeak

[3] similarly developed a hybrid block method for the direct numerical solution of first-order ordinary differential equations.

In parallel with these ordinary differential equation advancements, second-kind Volterra integral equations (VIEs) regularly emerge within mathematical modeling and must be treated numerically when analytical solutions are inaccessible. Classical quadrature, interpolation, and collocation techniques have been widely deployed, yet they suffer from severe instability when encountering nonlinear kernels and stiff systems. Due to their capacity to attain high-order precision with comparatively modest error constants, hybrid block methods and exponentially fitted approaches have recently been successfully extended to Volterra integral equations.

Other numerical treatments of Volterra integral equations include [4, 9] and [14], who combined quadrature rules for improved approximation. Raymond *et al*., (2023) contributed a series of hybrid and exponentially fitted block collocation methods specifically for second-kind Volterra integral equations, further reinforcing the relevance of hybrid block approaches to this class.

Despite these mathematical advancements, optimized hybrid block techniques created especially for the direct numerical solution of Volterra integral equations of the second kind—and their corresponding equivalent fourth-order ODE initial value problems (IVPs) have received very little attention in the literature. This is particularly true when it comes to the complex process of optimizing multiple hybrid points while incorporating higher-derivative information.

This study directly addresses this academic void by presenting an optimized two-step hybrid numerical block scheme incorporating a free parameter for the direct solution of second-kind Volterra integral equations. By building upon the structural equivalence between second-kind Volterra equations and fourth-order derivative systems, this research merges the principles of power series approximations and exponentially-fitted basis functions. Through a rigorous Taylor series error minimization process, one of the four off-grid hybrid points is optimized as a free parameter to provide an ideal parameter value

$\left(r = \frac{4}{400}\right)$, significantly enhancing accuracy, stability, and computational economy.

In order to solve volterra integral equations and starting value problems involving fourth derivative ordinary differential equations, this study focuses on an optimized numerical type hybrid Volterra integral equation with free parameter, which is expressed as

$$K(t) = \mu(t) + \int_{\rho(t)}^{\phi(t)} Q\,(t, v, K, K^{r}, K^{rr}, K^{rrr})\,dv \;\; \rho(t) \le v \le Q(t) \le X \qquad (1)$$

where the function μ(t) and Q(t, v, ϕ, ϕ^r, ϕ^rr, ϕ^rrr) are sufficiently smooth and has a unique solution K(t). Solving (1) is equivalent to solving the initial value problem for ordinary differential equations of the fourth derivative as follows:

$$K^{(iv)}(t) = \mu(t) + Q(t, K, K^{r}, K^{rr}, K^{rrr}),$$

$$K(t_0) = \mu(t_0), \quad K^{r}(t_0) = \mu^{r}(t_0), \quad K^{rr}(t_0) = \mu^{rr}(t_0), \quad K^{rrr}(t_0) = \mu^{rrr}(t_0) \qquad (2)$$

By including derivative information and carefully choosing hybrid points, the approach is designed to improve accuracy, stability, and computational economy. The resulting formulation is applied to both Volterra integral equations, and appropriate numerical examples are used to evaluate its performance. A helpful framework for creating direct numerical methods is provided by the equivalency between second-kind Volterra integral equations and their analogous higher-order differential equations. Imanova and Ibrahimov (2021) have conducted earlier research on numerical integration, quadrature, and coefficient calculation for such formulations. The current work builds on existing contributions by concentrating on an improved hybrid formulation that aims to enhance the numerical treatment of Volterra integral equations and fourth-order issues.

II. DERIVATION OF THE METHOD

To derive this method, four off-grid points will be introduced which are r_1, r_2, r_3, r_4 placed between t_n and t_{n+1} respectively. Therefore, we shall optimize r_1 and keep r_2, r_3, r_4 as free parameters.

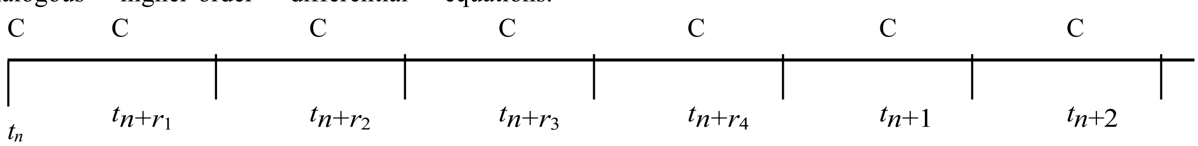

Figure 1: Collocation points

Let the approximate power series and exponentially fitted function be of the form

$$\sum_{i=0}^{k} \tau_i K_{n+i}^{(iv)} = \sum_{i=0}^{k} \tau_i \mu_{n+i} + h^4 \sum_{i=0}^{k} \zeta_i v_{n+i} + h^4 \sum_{j=1}^{r} \varphi_j e^{t_j} \qquad (3)$$

$$\sum_{i=0}^{k} \tau_i v_{n+i}^{(iv)} = h^4 \sum_{j=0}^{s} \zeta_j v_{n+i} + \sum_{j=1}^{r} \varphi_j e^{t_j} \quad (4)$$

where τ_i, ζ_j, φ_j (i = 0, 1, ….., m; j = 1, ….., k) are constant coefficients of (4) to be determined which is considered as the solution to the Volterra integral equation of the second kind given as

$$\sum_{i=0}^{m} \tau_i K_{n+i} - \mu_{n+i} = h^4 \sum_{i=0}^{k} \sum_{j=1}^{k} \varphi_i^{(j)} k_{t_{n+i}, t_{n+j}} K_{n+i} \qquad (5)$$

It is important to note that (5) can be written as

$$\sum_{i=0}^{k} \tau_i \alpha_{n+i} = h^4 \sum_{i=0}^{k} \sum_{j=1}^{k} \varphi_i^{(j)} k_{t_{n+i}, t_{n+j}} K_{n+i} \quad (6)$$

subject to the following condition

$$\alpha(t) = K(t) - \mu(t) \qquad (7)$$

With first, second, third and fourth derivative of equation (3) we obtain

$$K^{r}(t) - \mu^{r}(t) = \sum_{j=0}^{s} i\, t^{\,i-1} v_j + \sum_{j=1}^{4} i\, t^{\,i-1} \varphi_j e^{t_j} \quad (8)$$

$$K^{rr}(t) - \mu^{rr}(t) = \sum_{j=0}^{s} i\,(i-1) t^{\,i-2} v_j + \sum_{j=1}^{4} \varphi_j\,(i-1) i^{\,i-2} + j i^{2(i-2)} e^{t_j} \qquad (9)$$

$$K^{rrr}(t) - \mu^{rrr}(t) = \sum_{j=0}^{s} i\,(i-1)(i-2) t^{\,i-3} v_j(t) + \sum_{j=1}^{4} i\, t^{\,i-3} \varphi_j(t) e^{t_j} [i^2 e^{2j} - 3j - 3jtj + 3j^2 tj + j^2 + 2] \qquad (10)$$

$$K^{(4)}(t) - \mu^{4}(t) = \sum_{j=0}^{s} v_i \; i(j-1)(j-2)(j-3) t^{j-4} + \sum_{j=0}^{4} \varphi_i\, e^{jt}\, i(j-1)(j-2)(j-3) t^{j-4} + j^2(j-1)(7j-11) t^{2j-4} + 6j^3(j-1) t^{3j-4} \quad (11)$$

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Where $v_i, \varphi_i \in \mathbb{R}$ are real unknown values to be determined. We shall now consider four hybrid points r_1, r_2, r_3 and r_4 such that $0 < r_1 < r_2 < r_3 < r_4 < \cdots < N$ hold [1].

Interpolating (3), (8), (9) and (10) at the points $t_n = 0$ and collating (11) at all points $t_{n+m} = t_n + qh$, $m = \{0, r_1, r_2, r_3, r_4, 1,2\}$ yields a system of nonlinear equations.

$$AB = C \quad (12)$$

$$\begin{bmatrix} 24 & 24r_n & 18t_n^2 & 14t_n^3 & \frac{1+4}{2}t_n^4 & \frac{1+5}{10}t_n^5 & \frac{1+6}{60}t_n^6 & \frac{1+7}{420}t_n^7 & \frac{1+8}{3360}t_n^8 & \frac{1+9}{30240}t_n^9 \\ 0 & 24 & 36t_n & 42t_n^2 & 2t_n^3 & \frac{1+4}{2}t_n^4 & \frac{1+5}{10}t_n^5 & \frac{1+6}{60}t_n^6 & \frac{1+7}{420}t_n^7 & \frac{1+8}{3360}t_n^8 \\ 0 & 0 & 36 & 84t_n & 6t_n^2 & 2t_n^3 & \frac{1+4}{2}t_n^4 & \frac{1+5}{10}t_n^5 & \frac{1+6}{60}t_n^6 & \frac{1+7}{420}t_n^7 \\ 0 & 0 & 0 & 84 & 12t_n & 6t_n^2 & 2t_n^3 & \frac{1+4}{2}t_n^4 & \frac{1+5}{10}t_n^5 & \frac{1+6}{60}t_n^6 \\ 0 & 0 & 0 & 0 & 12 & 12t_n & 6t_n^2 & 2t_n^3 & \frac{1+4}{2}t_n^4 & \frac{1+5}{10}t_n^5 \end{bmatrix}$$

$$\frac{1+6}{60}t_n^6 \quad \frac{1+7}{420}t_n^7 \quad \frac{1+8}{3360}t_n^8 \quad \frac{1+9}{30240}t_n^9 \quad \frac{1+10}{302400}t_n^{10}$$

$$\begin{matrix} 0 & 0 & 0 & 0 & 12 & 12t_{n+r_1} & 6t_{n+r_1}^2 & 2t_{n+r_1}^3 & \frac{1+4}{2}t_{n+r_1}^4 & \frac{1+5}{10}t_{n+r_1}^5 \\ 0 & 0 & 0 & 0 & 12 & 12t_{n+r_2} & 6t_{n+r_2}^2 & 2t_{n+r_2}^3 & \frac{1+4}{2}t_{n+r_2}^4 & \frac{1+5}{10}t_{n+r_2}^5 \\ 0 & 0 & 0 & 0 & 12 & 12t_{n+r_3} & 6t_{n+r_3}^2 & 2t_{n+r_3}^3 & \frac{1+4}{2}t_{n+r_3}^4 & \frac{1+5}{10}t_{n+r_3}^5 \\ 0 & 0 & 0 & 0 & 12 & 12t_{n+r_4} & 6t_{n+r_4}^2 & 2t_{n+r_4}^3 & \frac{1+4}{2}t_{n+r_4}^4 & \frac{1+5}{10}t_{n+r_4}^5 \\ 0 & 0 & 0 & 0 & 12 & 12t_{n+1} & 6t_{n+1}^2 & 2t_{n+1}^3 & \frac{1+4}{2}t_{n+1}^4 & \frac{1+5}{10}t_{n+1}^5 \\ 0 & 0 & 0 & 0 & 12 & 12t_{n+2} & 6t_{n+2}^2 & 2t_{n+2}^3 & \frac{1+4}{2}t_{n+2}^4 & \frac{1+5}{10}t_{n+2}^5 \end{matrix}$$

$$\begin{bmatrix} v_i \\ v_{r_1} \\ v_{r_2} \\ v_{r_3} \\ v_{r_4} \\ v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} Q_n \\ Q_{r_1} \\ Q_{r_2} \\ Q_{r_3} \\ Q_{r_4} \\ Q_{n+1} \\ Q_{n+2} \end{bmatrix}$$

Using Gaussian elimination method on (12) gives the coefficients of $v_i, \varphi_i \ j = 0, r_1, r_2, r_3, r_4, 1,2, \ i = 0(1)3$.

These values are then substituted into (1) to give the implicit continuous optimized hybrid Volterra integral equation of the second kind scheme of the form:

$$p_i(K_n + nh) - (\mu_n + nh) = hv_0(K_n - \mu_n) + hv_1(K_n^r - \mu_n^r) + h^2 v_2(K_n^{rr} - \mu_n^{rr}) + h^3 v_3(K_n^{rrr} - \mu_n^{rrr}) + h^4 \varphi_0 Q_n + \varphi_{r_1} Q_{n+r_1} + \varphi_{r_2} Q_{n+r_2} + \varphi_{r_3} Q_{n+r_3} + \varphi_{r_4} Q_{n+r_4} + \varphi_1 Q_{n+1} + \varphi_2 Q_{n+2} \quad (13)$$

where: $v_0 = 1, \ v_1 = \eta, \ v_2 = \frac{1}{2}\eta^2, \ v_3 = \frac{1}{6}\eta^3$

$\varphi_0 =$

$$\frac{1}{30\cdot 240}\eta^4\left[\frac{\begin{array}{l}18\eta^4-15\eta^5+3\eta^6-36r_1\eta^3+27r_1\eta^4-5r_1\eta^5-36r_2\eta^3+27r_2\eta^4-5r_2\eta^5-36r_3\eta^3\\+27r_3\eta^4-5r_3\eta^5-36r_4\eta^3+27r_4\eta^4-5r_4\eta^5+84r_1r_2\eta^2-54r_1r_2\eta^3+9r_1r_2\eta^4\\+84r_1r_3\eta^2-54r_1r_3\eta^3+9r_1r_3\eta^4+84r_1r_4\eta^2-54r_1r_4\eta^3+9r_1r_4\eta^4+84r_2r_3\eta^2\\-54r_2r_3\eta^3+9r_2r_3\eta^4+84r_2r_4\eta^2-54r_2r_4\eta^3+9r_2r_4\eta^4+84r_3r_4\eta^2-54r_3r_4\eta^3\\+9r_3r_4\eta^4-252r_1r_2r_3\eta-252r_1r_2r_4\eta-252r_1r_3r_4\eta-252r_2r_3r_4\eta+126r_1r_2r_3\eta^2\\-18r_1r_2r_3\eta^3+126r_1r_2r_4\eta-18r_1r_2r_4\eta^3+126r_1r_3r_4\eta^2-18r_1r_3r_4\eta^3+126r_2r_3r_4\eta^2\\-18r_1r_2r_3r_4+1260r_1r_2r_3r_4-378r_1r_2r_3r_4\eta+42r_1r_2r_3r_4\eta^2\end{array}}{r_1r_2r_3r_4}\right]$$

$\varphi_{r_1} =$

$$\frac{1}{15\cdot 120}\eta^5\left[\frac{\begin{array}{l}18\eta^3-15\eta^4+3\eta^5-36r_2\eta^2+27r_2\eta^3-5r_2\eta^4-36r_3\eta^2+27r_3\eta^3-5r_3\eta^4-36r_4\eta^2\\+27r_4\eta^3-5r_4\eta^4+84r_2r_3\eta+84r_2r_4\eta+84r_3r_4\eta-54r_2r_3\eta^2+9r_2r_3\eta^3\\-54r_2r_4\eta^2+9r_2r_4\eta^3-54r_3r_4\eta^2+9r_3r_4\eta^3-252r_2r_3r_4+126r_2r_3r_4\eta-18r_2r_3r_4\eta^2\end{array}}{r_1(r_1-r_2)(r_1-r_3)(r_1-r_3)(r_1-1)(r_1-2)}\right]$$

$\varphi_{r_2} =$

$$-\frac{1}{15\cdot 120}\eta^5\left[\frac{\begin{array}{l}18\eta^3-15\eta^4+3\eta^5-36r_1\eta^2+27r_1\eta^3-5r_1\eta^4-36r_3\eta^2+27r_3\eta^3-5r_3\eta^4-36r_4\eta^2\\+27r_4\eta^3-5r_4\eta^4+84r_1r_3\eta+84r_1r_4\eta+84r_3r_4\eta-54r_1r_3\eta^2+9r_1r_3\eta^3\\-54r_1r_4\eta^2+9r_1r_4\eta^3-54r_3r_4\eta^2+9r_3r_4\eta^3-252r_1r_3r_4+126r_1r_3r_4\eta-18r_1r_3r_4\eta^2\end{array}}{r_2(r_1-r_2)(r_2-r_3)(r_2-r_4)(r_2-1)(r_2-2)}\right]$$

$\varphi_{r_3} =$

$$\frac{1}{15\cdot 120}\eta^5\left[\frac{\begin{array}{l}18\eta^3-15\eta^4+3\eta^5-36r_1\eta^2+27r_1\eta^3-5r_1\eta^4-36r_2\eta^2+27r_2\eta^3-5r_2\eta^4-36r_4\eta^2\\+27r_4\eta^3-5r_4\eta^4+84r_1r_2\eta+84r_1r_4\eta+84r_2r_4\eta-54r_1r_2\eta^2+9r_1r_2\eta^3\\-54r_1r_4\eta^2+9r_1r_4\eta^3-54r_2r_4\eta^2+9r_2r_4\eta^3-252r_1r_2r_4+126r_1r_2r_4\eta-18r_1r_2r_4\eta^2\end{array}}{r_3(r_1-r_3)(r_2-r_3)(r_3-r_4)(r_3-1)(r_3-2)}\right]$$

$\varphi_{r_4} =$

$$-\frac{1}{15\cdot 120}\eta^5\left[\frac{\begin{array}{l}18\eta^3-15\eta^4+3\eta^5-36r_1\eta^2+27r_1\eta^3-5r_1\eta^4-36r_2\eta^2+27r_2\eta^3-5r_2\eta^4-36r_3\eta^2\\+27r_3\eta^3-5r_3\eta^4+84r_1r_2\eta+84r_1r_3\eta+84r_2r_3\eta-54r_1r_2\eta^2+9r_1r_2\eta^3-54r_1r_3\eta^2\\+9r_1r_3\eta^3-54r_2r_3\eta^2+9r_2r_3\eta^3-252r_1r_2r_3+126r_1r_2r_3\eta-18r_1r_2r_3\eta^2\end{array}}{r_4(r_1-r_4)(r_2-r_4)(r_3-r_4)(r_4-1)(r_4-2)}\right]$$

$\varphi_1 =$

$$-\frac{1}{15\cdot 120}\eta^5\left[\frac{\begin{array}{l}3\eta^5-10\eta^4+18r_1\eta^3-5r_1\eta^4+18r_2\eta^3-5r_2\eta^4+18r_3\eta^3-5r_3\eta^4+18r_4\eta^3-5r_4\eta^4\\-36r_1r_2\eta^2+9r_1r_2\eta^3-36r_1r_3\eta^2+9r_1r_3\eta^3-36r_1r_4\eta^2+9r_1r_4\eta^3-36r_2r_3\eta^2\\+9r_2r_3\eta^3-36r_2r_4\eta^2+9r_2r_4\eta^3-36r_3r_4\eta^2+9r_3r_4\eta^3+84r_1r_2r_3\eta+84r_1r_2r_4\eta\\+84r_1r_3r_4\eta+84r_2r_3r_4\eta-18r_1r_2r_3\eta^2-18r_1r_2r_4\eta^2-18r_1r_3r_4\eta^2-18r_2r_3r_4\eta^2\\-252r_1r_2r_3r_4+42r_1r_2r_3r_4\eta\end{array}}{(r_1-1)(r_2-1)(r_3-1)(r_4-1)}\right]$$

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$\varphi_2 =$

$$\frac{1}{30\cdot 240}\eta^5\left[\frac{\begin{array}{l}3\eta^5-5\eta^4+9r_1\eta^3-5r_1\eta^4+9r_2\eta^3-5r_2\eta^4+9r_3\eta^3-5r_3\eta^4+9r_4\eta^3-5r_4\eta^4\\-18r_1r_2\eta^2+9r_1r_2\eta^3-18r_1r_3\eta^2+9r_1r_3\eta^3-18r_1r_4\eta^2+9r_1r_4\eta^3-18r_2r_3\eta^2\\+9r_2r_3\eta^3-18r_2r_4\eta^2+9r_2r_4\eta^3-18r_3r_4\eta^2+9r_3r_4\eta^3+42r_1r_2r_3\eta+42r_1r_2r_4\eta\\+42r_1r_3r_4\eta+42r_2r_3r_4\eta-18r_1r_2r_3\eta^2-18r_1r_2r_4\eta^2-18r_1r_3r_4\eta^2-18r_2r_3r_4\eta^2\\-126r_1r_2r_3r_4+42r_1r_2r_3r_4\eta\end{array}}{(r_1-2)(r_2-2)(r_3-2)(r_4-2)}\right]$$

Differentiating (13) once, twice and thrice we obtain

$$hv_1^{rr}(K_n+nh)-(\mu_n+nh)=hv_1(K_n^r-\mu_n^r)+h^2v_2(K_n^{rr}-\mu_n^{rr})+h^3v_3(K_n^{rrr}-\mu_n^{rrr})\ +h^4\varphi_0Q_n+\varphi_{r_1}Q_{n+r_1}+\varphi_{r_2}Q_{n+r_2}+\varphi_{r_3}Q_{n+r_3}+\varphi_{r_4}Q_{n+r_4}+\varphi_1Q_{n+1}+\varphi_2Q_{n+2} \qquad (14)$$

$$h^2v_2^{rrr}(K_n+nh)-(\mu_n+nh)=h^2v_2(K_n^{rr}-\mu_n^{rr})+h^3v_3(K_n^{rrr}-\mu_n^{rrr})\ +h^4\varphi_0Q_n+\varphi_{r_1}Q_{n+r_1}+\varphi_{r_2}Q_{n+r_2}+\varphi_{r_3}Q_{n+r_3}+\varphi_{r_4}Q_{n+r_4}+\varphi_1Q_{n+1}+\varphi_2Q_{n+2} \qquad (15)$$

$$h^3v_3^{rrrr}(K_n+nh)-(\mu_n+nh)=h^3v_3(K_n^{rrr}-\mu_n^{rrr})\ +h^4\varphi_0Q_n+\varphi_{r_1}Q_{n+r_1}+\varphi_{r_2}Q_{n+r_2}+\varphi_{r_3}Q_{n+r_3}+\varphi_{r_4}Q_{n+r_4}+\varphi_1Q_{n+1}+\varphi_2Q_{n+2} \qquad (16)$$

Substituting $\eta=1$ in (13) we obtain an approximate Volterra integral equation to the solution of (2) at the points t_{n+1} which yield

$$\rho(K_{n+1}h)-(\mu_{n+1}h)=(K_n-\mu_n)+hv_1(K_n^r-\mu_n^r)+\frac{1}{2}h^2v_2(K_n^{rr}-\mu_n^{rr})+\frac{1}{6}h^3v_3(K_n^{rrr}-\mu_n^{rrr})+$$
$$\frac{1}{30240}h^4\left[\frac{14(r_1+r_2+r_3+r_4)+39(r_1r_3+r_1r_4+r_2r_3+r_2r_4+r_3r_4)-144(r_1r_2r_3-r_1r_4-r_1r_3-r_2r_4-r_2r_3+r_3r_4)+6}{r_1r_2r_3r_4}\right]Q_n$$

$$-\frac{1}{15120}h^4\left[\frac{14(r_2+r_3+r_4)-39(r_2r_3+r_2r_4+r_3r_4)+144r_2r_3r_4+6}{r_1(r_1-1)(r_1-2)(r_1-r_2)(r_1-r_3)(r_1-r_4)}\right]Q_{n+r_1}+$$
$$\frac{1}{15120}h^4\left[\frac{14(r_1+r_3+r_4)-39(r_1r_3+r_1r_4+r_3r_4)+144r_1r_3r_4+6}{r_2(r_2-1)(r_2-2)(r_2-r_1)(r_2-r_3)(r_2-r_4)}\right]Q_{n+r_2}\ +$$
$$h^3\left[-\frac{1}{15120}\frac{14(r_1+r_2+r_4)-39(r_1r_2+r_1r_4+r_2r_4)+144r_1r_2r_4+6}{r_3(r_3-1)(r_3-2)(r_3-r_1)(r_3-r_2)(r_3-r_4)}\right]Q_{n+r_3}\ +$$
$$\frac{1}{15120}\left[\frac{14r_2+14r_3+14r_1-39r_2r_3-39r_2r_1-39r_3r_1+144r_2r_3r_1+6}{r_4(r_4-1)(r_4-2)(r_4-r_1)(r_4-r_2)(r_4-r_3)}\right]Q_{n+r_4}\ +$$
$$\frac{1}{15120}\left[\frac{-13(r_1+r_2+r_3+r_4)+27(r_1r_2+r_1r_3+r_1r_4+r_2r_3+r_2r_4+r_3r_4)-66(r_1r_2r_3+r_1r_2r_4+r_1r_3r_4+r_2r_3r_4)+210r_1r_2r_3r_4+7}{(r_4-1)(r_2-1)(r_3-1)(r_1-1)}\right]Q_{n+1}$$

$$-\frac{1}{30240}\left[\frac{-4(r_1+r_2+r_3+r_4)+9(r_1r_2+r_1r_3+r_1r_4+r_2r_3+r_2r_4+r_3r_4)-24(r_1r_2r_3+r_1r_2r_4+r_1r_3r_4+r_2r_3r_4)+84r_1r_2r_3r_4+2}{(r_4-2)(r_2-2)(r_3-2)(r_1-2)}\right]Q_{n+2}$$
(17)

Also substituting $\eta=1$ in (14) we obtain an approximate Volterra integral equation to the solution of (2) at the points t_{n+1} which yield

$$h\rho^r(K_n+h)-(\mu_n+h)=hv_1(K_n^r-\mu_n^r)+\frac{1}{2}h^2v_2(K_n^{rr}-\mu_n^{rr})+\frac{1}{6}h^3v_3(K_n^{rrr}-\mu_n^{rrr})\ -$$
$$\frac{1}{10080}h^4\left[\frac{-27(r_1-r_2-r_3-r_4)+66(r_1r_2+r_1r_3+r_1r_4+r_2r_3+r_2r_4+r_3r_4)-210(r_1r_2r_3+r_1r_2r_4+r_1r_3r_4+r_2r_3r_4)+1134r_1r_2r_3r_4+13}{r_1r_2r_3r_4}\right]Q_n$$

$$-\frac{1}{5040}h^4\left[\frac{27(r_2+r_3+r_4)-66(r_2r_3+r_2r_4+r_3r_4)-210r_2r_3r_4-13}{r_1(r_1-1)(r_1-2)(r_1-r_2)(r_1-r_3)(r_1-r_4)}\right]Q_{n+r_1}$$

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$$+\frac{1}{5040}h^4\left[\frac{27r_1+27r_3+27r_4-66r_1r_3-66r_1r_4-66r_3r_4+210r_1r_3r_4-13}{r_2(r_2-1)(r_2-2)(r_2-r_1)(r_2-r_3)(r_2-r_4)}\right]Q_{n+r_2}$$
$$+h^3\left[-\frac{1}{5040}\frac{27(r_1+r_2+r_4)-66(r_1r_2+r_1r_4+r_2r_4)+210r_1r_2r_4-13}{r_3(r_3-1)(r_3-2)(r_3-r_1)(r_3-r_2)(r_3-r_4)}\right]Q_{n+r_3}$$
$$+\frac{1}{5040}\left[\frac{27(r_1+r_2+r_3)-66(r_1r_2+r_1r_3+r_2r_3)+210r_1r_2r_3-13}{r_4(r_4-1)(r_4-2)(r_4-r_1)(r_4-r_2)(r_4-r_3)}\right]Q_{n+r_4}$$
$$-\frac{1}{5040}\left[\frac{-13(r_1+r_2+r_3+r_4)+60(r_1r_2+r_1r_3+r_1r_4+r_2r_3+r_2r_4+r_3r_4)-126r_1r_2r_3r_4}{(r_4-1)(r_2-1)(r_3-1)(r_1-1)}\right]Q_{n+1}$$
$$-\frac{1}{10080}\left[\frac{-9(r_1+r_2+r_3+r_4)+18(r_1r_2+r_1r_3+r_1r_4+r_2r_3+r_2r_4+r_3r_4)-42(r_1r_2r_3+r_1r_2r_4+r_1r_3r_4+r_2r_3r_4)+126r_1r_2r_3r_4+5}{(r_4-2)(r_2-2)(r_3-2)(r_1-2)}\right]Q_{n+2} \quad (18)$$

Also substituting $\eta = 1$ in (15) we obtain an approximate Volterra integral equation to the solution of (2) at the points t_{n+1} which yield

$$h^2\rho^{rr}(K_n+h)-(\mu_n+h)=\frac{1}{2}h^2v_2(K_n^{rr}-\mu_n^{rr})+\frac{1}{6}h^3v_3(K_n^{rrr}-\mu_n^{rrr})$$

$$-\frac{1}{1680}h^4\left[\frac{-20(r_1+r_2+r_3+r_4)+42(r_1r_2+r_1r_3+r_1r_4+r_2r_3+r_2r_4+r_3r_4)-112(r_1r_2r_3+r_1r_2r_4+r_1r_3r_4+r_2r_3r_4)+490r_1r_2r_3r_4}{r_1r_2r_3r_4}\right]Q_n$$

$$-\frac{1}{840}h^4\left[\frac{20(r_2+r_3+r_4)-42(r_2r_3+r_2r_4+r_3r_4)+112r_2r_3r_4-11}{r_1(r_1-1)(r_1-2)(r_1-r_2)(r_1-r_3)(r_1-r_4)}\right]Q_{n+r_1}$$
$$+\frac{1}{840}h^4\left[\frac{20(r_1+r_3+r_4)-42(r_1r_3+r_1r_4+r_3r_4)+112r_1r_3r_4-11}{r_2(r_2-1)(r_2-2)(r_2-r_1)(r_2-r_3)(r_2-r_4)}\right]Q_{n+r_2}$$
$$+h^4\left[-\frac{1}{840}\frac{20(r_1+r_2+r_4)-42(r_1r_2+r_1r_4+r_2r_4)+112r_1r_2r_4-11}{r_3(r_3-1)(r_3-2)(r_3-r_1)(r_3-r_2)(r_3-r_4)}\right]Q_{n+r_3}$$
$$+\frac{1}{840}h^4\left[\frac{20(r_1+r_2+r_3)-42(r_1r_2+r_1r_3+r_2r_3)+112r_1r_2r_3-11}{r_4(r_4-1)(r_4-2)(r_4-r_1)(r_4-r_2)(r_4-r_3)}\right]Q_{n+r_4}$$
$$+\frac{1}{840}h^4\left[\frac{-36r_1-36r_2-36r_3-36r_4+54r_1r_2+54r_1r_3+54r_1r_4+54r_2r_3+54r_2r_4+54r_3r_4-98r_1r_2r_3-98r_1r_2r_4-98r_1r_3r_4-98r_2r_3r_4+210r_1r_2r_3r_4+25}{(r_4-1)(r_2-1)(r_3-1)(r_1-1)}\right]Q_{n+1}$$
$$-\frac{1}{840}h^4\left[\frac{-8(r_1+r_2+r_3+r_4)+14(r_1r_2+r_1r_3+r_1r_4+r_2r_3+r_2r_4+r_3r_4)-28(r_1r_2r_3+r_1r_2r_4+r_1r_3r_4+r_2r_3r_4)+70r_1r_2r_3r_4+5}{(r_4-2)(r_2-2)(r_3-2)(r_1-2)}\right]Q_{n+2} \quad (19)$$

Also substituting $\eta = 1$ in (16) we obtain an approximate Volterra integral equation to the solution of (2) at the points t_{n+1} which yield

$$h^3\rho^{rrr}(K_n+h)-(\mu_n+h)=h^3v_3(K_n^{rrr}-\mu_n^{rrr})$$
$$-\frac{1}{840}h^4\left[\frac{-28(r_1+r_2+r_3+r_4)+49(r_1r_2+r_1r_3+r_1r_4+r_2r_3+r_2r_4+r_3r_4)-105(r_1r_2r_3+r_1r_2r_4+r_1r_3r_4+r_2r_3r_4)+350r_1r_2r_3r_4+18}{r_1r_2r_3r_4}\right]Q_n$$

$$+h^4\left[-\frac{1}{420}\frac{28(r_2+r_3+r_4)-49(r_2r_3+r_2r_4+r_3r_4)+105r_2r_3r_4-18}{r_1(r_1-1)(r_1-2)(r_1-r_2)(r_1-r_3)(r_1-r_4)}\right]Q_{n+r_1}$$
$$+\frac{1}{420}h^4\left[\frac{28r_1+28r_3+28r_4-49r_1r_3-49r_1r_4-49r_3r_4+105r_1r_3r_4-18}{r_2(r_2-1)(r_2-2)(r_2-r_1)(r_2-r_3)(r_2-r_4)}\right]Q_{n+r_2}$$
$$+h^4\left[-\frac{1}{420}\frac{28(r_1+r_2+r_4)-49(r_1r_2+r_1r_4+r_2r_4)+105r_1r_2r_4-18}{r_3(r_3-1)(r_3-2)(r_3-r_1)(r_3-r_2)(r_3-r_4)}\right]Q_{n+r_3}$$
$$+\frac{1}{420}h^4\left[\frac{20(r_1+r_2+r_3)-42(r_1r_2+r_1r_3+r_2r_3)+112r_1r_2r_3-11}{r_4(r_4-1)(r_4-2)(r_4-r_1)(r_4-r_2)(r_4-r_3)}\right]Q_{n+r_4}$$
$$+\frac{1}{15120}h^4\left[\frac{-98(r_1+r_2+r_3+r_4)+126(r_1r_2+r_1r_3+r_1r_4+r_2r_3+r_2r_4+r_3r_4)-175(r_1r_2r_3+r_1r_2r_4+r_1r_3r_4+r_2r_3r_4)+280r_1r_2r_3r_4+80}{(r_4-1)(r_2-1)(r_3-1)(r_1-1)}\right]Q_{n+1}$$

$$-\frac{1}{840}h^4\left[\frac{-14(r_1+r_2+r_3+r_4)+21(r_1r_2+r_1r_3+r_1r_4+r_2r_3+r_2r_4+r_3r_4)-35(r_1r_2r_3+r_1r_2r_4+r_1r_3r_4+r_2r_3r_4)+70r_1r_2r_3r_4+10}{(r_4-2)(r_2-2)(r_3-2)(r_1-2)}\right]Q_{n+2} \quad (20)$$

The following steps are taking to obtain the optimal value of r

✓ Expand (18) by Taylor's series to obtain the corresponding local truncation error as

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$$L[K'(t_{n+1});h] = \frac{-36r_1-36r_2-36r_3-36r_4+56r_1r_2+56r_1r_3+56r_1r_4+56r_2r_3+56r_2r_4+56r_3r_4-98r_1r_2r_3-98r_1r_2r_4-98r_1r_3r_4-98r_2r_3r_4+210r_1r_2r_3r_4+25}{176400}$$

✓ Equate the principal term of truncation error to zero by keeping r_2, r_3 and r_4 as free parameter by assigning $r_2 = \frac{2}{5}, r_3 = \frac{3}{5}, r_4 = \frac{4}{5}$

✓ Equate the principal term of truncation error to zero by keeping r_2, r_3 and r_4 as free parameter by assigning $r_2 = \frac{2}{5}, r_3 = \frac{3}{5}, r_4 = \frac{4}{5}$ to obtain $r_1 = \frac{47}{400}$

✓ Substitute the optimal value of r_1 and the free parameter values r_2, r_3 and r_4 into (17) - (20), we then obtain the discrete scheme of the Volterra type as shown below

$$K_{n+1} - \mu_{n+1} = (K_n - \mu_n) + \frac{47}{400}h(K_n^r - \mu_n^r) + \frac{2209}{320000}h^2(K_n^{rr} - \mu_n^{rr}) + \frac{103823}{384000000}h^3(K_n^{rrr} - \mu_n^{rrr}) + \frac{4234581519019588965073}{7610145177600000000000000}h^4Q_n + \frac{6141998581301264388833}{2041919341459845120000000000}h^4Q_{n+r_1} - \frac{9587128853440411635809}{6879571240550400000000000}h^4Q_{n+r_2}$$

$$- \frac{6207934605031450379329}{5140563506746880000000000}h^4Q_{n+r_3} - \frac{170040780912925718387}{307788095849600000000000}h^4Q_{n+r_4} + \frac{3642682464118674919169}{3357976559616000000000000}h^4Q_{n+1} - \frac{597374797995162878923}{2674205015408640000000000}h^4Q_{n+2}$$

$$K_{n+\frac{2}{5}} - \mu_{n+\frac{2}{5}} = (K_n - \mu_n) + \frac{2}{5}h(K_n^r - \mu_n^r) + \frac{2}{25}h^2(K_n^{rr} - \mu_n^{rr}) + \frac{4}{375}h^3(K_n^{rrr} - \mu_n^{rrr}) + \frac{732748}{2081953125}h^4Q_n$$

$$+ \frac{338899763200}{4686045363701793}h^4Q_{n+\frac{47}{400}} - \frac{368}{8008875}h^4Q_{n+\frac{2}{5}} + \frac{149608}{2393803125}h^4Q_{n+\frac{3}{5}} - \frac{188}{5971875}h^4Q_{n+\frac{4}{5}} + \frac{14408}{2233828125}h^4Q_{n+1} - \frac{1102}{77829609375}h^4Q_{n+2}$$

$$K_{n+\frac{3}{5}} - \mu_{n+\frac{3}{5}} = (K_n - \mu_n) + \frac{3}{5}h(K_n^r - \mu_n^r) + \frac{9}{50}h^2(K_n^{rr} - \mu_n^{rr}) + \frac{9}{250}h^3(K_n^{rrr} - \mu_n^{rrr}) + \frac{7801569}{6580000000}h^4Q_n$$

$$- \frac{208278650880}{57852411897553}h^4Q_{n+\frac{47}{400}} + \frac{952047}{1265600000}h^4Q_{n+\frac{2}{5}} - \frac{84321}{472850000}h^4Q_{n+\frac{3}{5}} + \frac{23391}{509600000}h^4Q_{n+\frac{4}{5}} - \frac{8721}{1544375000}h^4Q_{n+1} + \frac{533}{49196000000}h^4Q_{n+2}$$

$$K_{n+\frac{4}{5}} - \mu_{n+\frac{4}{5}} = (K_n - \mu_n) + \frac{4}{5}h(K_n^r - \mu_n^r) + \frac{8}{25}h^2(K_n^{rr} - \mu_n^{rr}) + \frac{32}{375}h^3(K_n^{rrr} - \mu_n^{rrr}) + \frac{5707904}{2081953125}h^4Q_n$$

$$+ \frac{48790828482560}{4686045363701793}h^4Q_{n+\frac{47}{400}} + \frac{895424}{200221875}h^4Q_{n+\frac{2}{5}} - \frac{2142208}{2393803125}h^4Q_{n+\frac{3}{5}} + \frac{21664}{53746875}h^4Q_{n+\frac{4}{5}} - \frac{1052672}{15636796875}h^4Q_{n+1} + \frac{7744}{77829609375}h^4Q_{n+2}$$

$$(K_{n+1} - \mu_{n+1}) = (K_n - \mu_n) + h(K_n^r - \mu_n^r) + \frac{1}{2}h^2(K_n^{rr} - \mu_n^{rr}) + \frac{1}{6}h^3(K_n^{rrr} - \mu_n^{rrr}) + \frac{178207}{34110720}h^4Q_n$$

$$+ \frac{107315200000000}{4686045363701793}h^4Q_{n+\frac{47}{400}} + \frac{63175}{4686336}h^4Q_{n+\frac{2}{5}} - \frac{7475}{6128136}h^4Q_{n+\frac{3}{5}} + \frac{6575}{4402944}h^4Q_{n+\frac{4}{5}} - \frac{683}{3202416}h^4Q_{n+1} + \frac{871}{2550320640}h^4Q_{n+2}$$

$$(K_{n+2} - \mu_{n+2}) = (K_n - \mu_n) + 2h(K_n^r - \mu_n^r) + 2h^2(K_n^{rr} - \mu_n^{rr}) + \frac{4}{3}h^3(K_n^{rrr} - \mu_n^{rrr}) - \frac{2164}{139245}h^4Q_n$$

$$+\frac{1782579200000000}{4686045363701793}h^4Q_{n+\frac{47}{400}}-\frac{11450}{64071}h^4Q_{n+\frac{2}{5}}+\frac{545800}{766017}h^4Q_{n+\frac{3}{5}}-\frac{8300}{17199}h^4Q_{n+\frac{4}{5}}+\frac{250744}{1000755}h^4Q_{n+1}-\frac{3628}{4981095}h^4Q_{n+2}$$

$$K^r_{n+\frac{47}{400}}-\mu^r_{n+\frac{47}{400}}=(K^r_n-\mu^r_n)+\frac{47}{400}h(K^{rr}_n-\mu^{rr}_n)+\frac{2209}{320000}h^2(K^{rrr}_n-\mu^{rrr}_n)+\frac{468798409039555566007}{253671505920000000000}h^3Q_n$$

$$+\frac{6178820949955775039}{5104798353649612800000}h^3Q_{n+\frac{47}{400}}-\frac{1193414553795876237 59}{22931904135168000000000}h^3Q_{n+\frac{2}{5}}+\frac{10982429428021455209}{24479300321280000000000}h^3Q_{n+\frac{3}{5}}$$

$$-\frac{5671793247402918820 7}{277009284464600000000000}h^3Q_{n+\frac{4}{5}}+\frac{6419813060146538613}{1599064569600000000000}h^3Q_{n+1}-\frac{3149987903941447673}{382029287915520000000000}h^3Q_{n+2}$$

$$K^r_{n+\frac{2}{5}}-\mu^r_{n+\frac{2}{5}}=(K^r_n-\mu^r_n)+\frac{2}{5}h(K^{rr}_n-\mu^{rr}_n)+\frac{2}{25}h^2(K^{rrr}_n-\mu^{rrr}_n)+\frac{375574}{138796875}h^3Q_n+\frac{268435456000}{360465027977061}h^3Q_{n+\frac{47}{400}}$$

$$+\frac{30077}{53392500}h^3Q_{n+\frac{2}{5}}-\frac{344}{22798125}h^3Q_{n+\frac{3}{5}}-\frac{122}{2480625}h^3Q_{n+\frac{4}{5}}+\frac{15448}{1042453125}h^3Q_{n+1}-\frac{421}{8894812500}h^3Q_{n+2}$$

$$K^r_{n+\frac{3}{5}}-\mu^r_{n+\frac{3}{5}}=(K^r_n-\mu^r_n)+\frac{3}{5}h(K^{rr}_n-\mu^{rr}_n)+\frac{9}{50}h^2(K^{rrr}_n-\mu^{rrr}_n)+\frac{7632927}{1316000000}h^3Q_n+\frac{26895974400}{1180661467297}h^3Q_{n+\frac{47}{400}}$$

$$+\frac{67149}{7232000}h^3Q_{n+\frac{2}{5}}-\frac{504909}{189140000}h^3Q_{n+\frac{3}{5}}+\frac{14067}{14560000}h^3Q_{n+\frac{4}{5}}-\frac{206637}{1235500000}h^3Q_{n+1}+\frac{23949}{98392000000}h^3Q_{n+2}$$

$$K^r_{n+\frac{4}{5}}-\mu^r_{n+\frac{4}{5}}=(K^r_n-\mu^r_n)+\frac{4}{5}h(K^{rr}_n-\mu^{rr}_n)+\frac{8}{25}h^2(K^{rrr}_n-\mu^{rrr}_n)+\frac{1578232}{138796875}h^3Q_n+\frac{219258080460800}{4686045363701793}h^3Q_{n+\frac{47}{400}}$$

$$+\frac{398968}{13348125}h^3Q_{n+\frac{2}{5}}-\frac{5328}{911925}h^3Q_{n+\frac{3}{5}}+\frac{92216}{32248125}h^3Q_{n+\frac{4}{5}}-\frac{72448}{148921875}h^3Q_{n+1}+\frac{1768}{2223703125}h^3Q_{n+2}$$

$$K^r_{n+1}-\mu^r_{n+1}=(K^r_n-\mu^r_n)+h(K^{rr}_n-\mu^{rr}_n)+\frac{1}{2}h^2(K^{rrr}_n-\mu^{rrr}_n)+\frac{171377}{11370240}h^3Q_n+\frac{409600000000}{51495003996723}h^3Q_{n+\frac{47}{400}}$$

$$+\frac{680125}{10934784}h^3Q_{n+\frac{2}{5}}-\frac{10625}{8170848}h^3Q_{n+\frac{3}{5}}+\frac{1375}{145152}h^3Q_{n+\frac{4}{5}}-\frac{9827}{10674720}h^3Q_{n+1}+\frac{4217}{2550320640}h^3Q_{n+2}$$

$$K^r_{n+2}-\mu^r_{n+2}=(K^r_n-\mu^r_n)+2h(K^{rr}_n-\mu^{rr}_n)+2h^2(K^{rrr}_n-\mu^{rrr}_n)-\frac{11966}{44415}h^3Q_n+\frac{838860800000000}{669435051957399}h^3Q_{n+\frac{47}{400}}$$

$$-\frac{23525}{12204}h^3Q_{n+\frac{2}{5}}+\frac{1011800}{255339}h^3Q_{n+\frac{3}{5}}-\frac{21850}{7371}h^3Q_{n+\frac{4}{5}}+\frac{424888}{333585}h^3Q_{n+1}-\frac{106553}{19924380}h^3Q_{n+2}$$

$$K^{rr}_{n+\frac{47}{400}}-\mu^{rr}_{n+\frac{47}{400}}=(K^{rr}_n-\mu^{rr}_n)+\frac{47}{400}h(K^{rrr}_n-\mu^{rrr}_n)+\frac{79769479154421437}{211392921600000000}h^2Q_n$$

$$+\frac{65099568217781}{17215696592640000}h^2Q_{n+\frac{47}{400}}-\frac{1361261692968304513}{955496005632000000000}h^2Q_{n+\frac{2}{5}}+\frac{870349159412249857}{713979592704000000000}h^2Q_{n+\frac{3}{5}}$$

$$-\frac{914165771536392 23}{16488647884800000000}h^2Q_{n+\frac{4}{5}}+\frac{57338872138697}{528482304000000000}h^2Q_{n+1}-\frac{4969610798428749}{222850417950720000000}h^2Q_{n+2}$$

$$K^{rr}_{n+\frac{3}{5}}-\mu^{rr}_{n+\frac{3}{5}}=(K^{rr}_n-\mu^{rr}_n)+\frac{2}{5}h(K^{rrr}_n-\mu^{rrr}_n)+\frac{80119}{6168750}h^2Q_n+\frac{85328920576000}{1562015121233931}h^2Q_{n+\frac{47}{400}}$$

$$+\frac{249}{14125}h^2Q_{n+\frac{2}{5}}-\frac{1048}{141855}h^2Q_{n+\frac{3}{5}}+\frac{11083}{4299750}h^2Q_{n+\frac{4}{5}}-\frac{3352}{7721875}h^2Q_{n+1}+\frac{688}{1037728125}h^2Q_{n+2}$$

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$$K^{rr}_{n+\frac{3}{5}} - \mu^{rr}_{n+\frac{3}{5}} = (K^{rr}_n - \mu^{rr}_n) + \frac{3}{5}h(K^{rrr}_n - \mu^{rrr}_n) + \frac{1188003}{65800000}h^2Q_n + \frac{5691408384000}{57852411897553}h^2Q_{n+\frac{47}{400}} + \frac{922655}{126560000}h^2Q_{n+\frac{2}{5}}$$

$$-\frac{2526}{168875}h^2Q_{n+\frac{3}{5}} + \frac{34917}{5096000}h^2Q_{n+\frac{4}{5}} - \frac{19017}{15443750}h^2Q_{n+1} + \frac{1473}{702800000}h^2Q_{n+2}$$

$$K^{rr}_{n+\frac{4}{5}} - \mu^{rr}_{n+\frac{4}{5}} = (K^{rr}_n - \mu^{rr}_n) + \frac{4}{5}h(K^{rrr}_n - \mu^{rrr}_n) + \frac{214336}{9253125}h^2Q_n + \frac{31675383808000}{223145017319133}h^2Q_{n+\frac{47}{400}} + \frac{23524}{177975}h^2Q_{n+\frac{2}{5}}$$

$$+\frac{103552}{10639125}h^2Q_{n+\frac{3}{5}} + \frac{4664}{307125}h^2Q_{n+\frac{4}{5}} - \frac{153728}{69496875}h^2Q_{n+1} + \frac{3748}{1037728125}h^2Q_{n+2}$$

$$K^{rr}_{n+1} - \mu^{rr}_{n+1} = (K^{rr}_n - \mu^{rr}_n) + h(K^{rrr}_n - \mu^{rrr}_n) + \frac{1795}{63168}h^2Q_n + \frac{819200000000}{4424972014827}h^2Q_{n+\frac{47}{400}} + \frac{19475}{101248}h^2Q_{n+\frac{2}{5}}$$

$$+\frac{2075}{56742}h^2Q_{n+\frac{3}{5}} + \frac{63325}{1100736}h^2Q_{n+\frac{4}{5}} + \frac{187}{42505344}h^2Q_{n+2}$$

$$K^{rr}_{n+2} - \mu^{rr}_{n+2} = (K^{rr}_n - \mu^{rr}_n) + 2h(K^{rrr}_n - \mu^{rrr}_n) + \frac{8699}{5922}h^2Q_n + \frac{7025459200000000}{1562015121233931}h^2Q_{n+\frac{47}{400}} - \frac{72650}{7119}h^2Q_{n+\frac{2}{5}} + \frac{213800}{12159}h^2Q_{n+\frac{3}{5}}$$

$$-\frac{465925}{54398}h^2Q_{n+\frac{4}{5}} + \frac{113480}{22239}h^2Q_{n+1} + \frac{1661}{47439}h^2Q_{n+2}$$

$$K^{rrr}_{n+\frac{47}{400}} - \mu^{rrr}_{n+\frac{47}{400}} = (K^{rrr}_n - \mu^{rrr}_n) + \frac{2977172493820447}{6606028800000000000}hQ_n + \frac{1104090652402973}{13293745712629200}hQ_{n+\frac{47}{400}} - \frac{13715059427301167}{597185003520000000}hQ_{n+\frac{2}{5}}$$

$$+\frac{1733191553030099}{89247449088000000}hQ_{n+\frac{3}{5}} - \frac{6337502302227983}{721378344960000000}hQ_{n+\frac{4}{5}} + \frac{4996071535045631}{2914910208000000000}hQ_{n+1} - \frac{2427994853956271}{6964075560960000000}hQ_{n+2}$$

$$K^{rrr}_{n+\frac{2}{5}} - \mu^{rrr}_{n+\frac{2}{5}} = (K^{rrr}_n - \mu^{rrr}_n) + \frac{59203}{2467500}hQ_n + \frac{348630548480000}{1562015121233931}hQ_{n+\frac{47}{400}} + \frac{99229}{474600}hQ_{n+\frac{2}{5}} - \frac{57464}{709275}hQ_{n+\frac{3}{5}}$$

$$+\frac{10333}{343980}hQ_{n+\frac{4}{5}} - \frac{24536}{4633125}hQ_{n+1} + \frac{14999}{1660365000}hQ_{n+2}$$

$$K^{rrr}_{n+\frac{3}{5}} - \mu^{rrr}_{n+\frac{3}{5}} = (K^{rrr}_n - \mu^{rrr}_n) + \frac{59203}{2467500}hQ_n + \frac{348630548480000}{1562015121233931}hQ_{n+\frac{47}{400}} + \frac{99229}{474600}hQ_{n+\frac{2}{5}} - \frac{57464}{709275}hQ_{n+\frac{3}{5}}$$

$$+\frac{10333}{343980}hQ_{n+\frac{4}{5}} - \frac{24536}{4633125}hQ_{n+1} + \frac{14999}{1660365000}hQ_{n+2}$$

$$K^{rrr}_{n+\frac{4}{5}} - \mu^{rrr}_{n+\frac{4}{5}} = (K^{rrr}_n - \mu^{rrr}_n) + \frac{45422}{1850625}hQ_n + \frac{344939560960000}{1562015121233931}hQ_{n+\frac{47}{400}} + \frac{50648}{177975}hQ_{n+\frac{2}{5}} - \frac{364736}{2127825}hQ_{n+\frac{3}{5}}$$

$$+\frac{9146}{85995}hQ_{n+\frac{4}{5}} - \frac{107456}{13899375}hQ_{n+1} + \frac{2096}{207545625}hQ_{n+2}$$

$$K^{rrr}_{n+1} - \mu^{rrr}_{n+1} = (K^{rrr}_n - \mu^{rrr}_n) + \frac{1795}{63168}hQ_n + \frac{327680000000000}{1562015121233931}hQ_{n+\frac{47}{400}} + \frac{97375}{303744}hQ_{n+\frac{2}{5}} + \frac{10375}{113484}hQ_{n+\frac{3}{5}}$$

$$+\frac{316625}{1100736}hQ_{n+\frac{4}{5}} + \frac{1843}{29652}hQ_{n+1} + \frac{187}{42505344}hQ_{n+2}$$

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$$K_{n+2}^{rrr} - \mu_{n+2}^{rrr} = (K_n^{rrr} - \mu_n^{rrr}) - \frac{55207}{11844} hQ_n + \frac{19922944000000000}{1562015121233931} hQ_{n+\frac{47}{400}} - \frac{1816625}{56952} hQ_{n+\frac{2}{5}} + \frac{4391000}{85113} hQ_{n+\frac{3}{5}}$$

$$+ \frac{299512}{22239} hQ_{n+1} + \frac{534825}{2656584} hQ_{n+2} \qquad (21)$$

III. ANALYSIS OF BASIC PROPERTIES OF THE METHOD

3.1 Order of the Block

Let the linear difference operator L associated with the new method be defined by

$$L[(K(t);h)] = A^{(0)} K_n^i - \mu_n^i - \sum_{j=0}^{3} h^j \alpha_j (K_n^i - \mu_n^i) - h^4 \sum_{j=0}^{2} \varphi_j Q_j (K_n - \mu_n) + \sum_{j=\eta}^{2} \varphi_j Q_j \ (22)$$

Where $\eta = \frac{47}{400}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}$

$K(x) - \mu(x)$ is the exact solution satisfying equation (22). It can be written in Taylor's series expansion about the point t_n to obtain the expression

$$L\big[\big(K(t) - \mu(t)\big); h\big] = c_0\big(K(t) - \mu(t)\big) + c_1 h\big(K^r(t) - \mu^r(t)\big) + c_2 h^2\big(K^{rr}(t) - \mu^{rr}(t)\big)$$
$$+ c_3 h^3\big(K^{rrr}(t) - \mu^{rrr}(t)\big) + \cdots + c_{p+3} h^{p+3} \frac{K^{(p+3)}(t) - \mu^{(p+3)}(t)}{(p+3)!} + \cdots \qquad (23)$$

Similarly, the Two-Step Hybrid Optimized Volterra Integral Equation of the second kind is of order p if,

$$L\big[\big(K(t) - \mu(t)\big); h\big] = O(h^{p+4}), \qquad \bar{c}_0 = \bar{c}_1 = \bar{c}_2 = \bar{c}_3 = \cdots = \bar{c}_{p+3} = 0, \qquad \bar{c}_{p+4} \neq 0$$

$$\bar{c}_{p+4} h^{p+4} (K - \mu)^{(p+4)}(t_n)$$

$C_{p+3} = 10$, in which its order is

$p = (7,7,7,7,7,7,7,7,7,7,7,7,7,7,7,7,7,7,7,7,7,7,7,8,7)^T$,

with error constant

$$C_{p+3} = \Big[\frac{110888390638130350872053 3}{1253675954304000000000000000} - \frac{149879}{255814453125000} - \frac{28773}{962500000}$$

$$- \frac{31264}{114202880859375} - \frac{7297}{320928000000} - \frac{2417}{334125000} - \frac{2886739333211594365515}{8878502707200000000000000} - \frac{9047}{193798828125}$$

$$- \frac{1225000000000}{828590337134011038850757} - \frac{538390078125}{359827} - \frac{1334}{30077} - \frac{206718750}{168303}$$

$$- \frac{17349474013619200000000000}{11527} - \frac{359827}{968994140625} - \frac{106553}{508032000000} - \frac{15805306250000}{108006693630124571}$$

$$- \frac{245000000000}{83} - \frac{421}{2734375000} - \frac{4961250000}{21875000000} - \frac{797273456000000000000}{583} - \frac{9827}{18457031250} - \frac{1661}{508032000000} - \frac{23625000}{}\Big]^T$$

hence, it is of order seven.

3.2 Zero Stability

The novel two-step hybrid optimizes Volterra Integral Equation of the second Kind (21), utilizing four off-grid collocation points articulated in the manner (22) to provide

$$\rho(z) = \det\left[\sum_{j=0}^{k} A_j\, z^{k-j}\right] = 0$$

satisfy $|z| \leq 1$ and for those roots with $z_i = 1$ multiplicity must not exceed two. Hence the method is to be stable.

3.3 Convergence

Theorem: *(Henrici, 1962)*

The necessary and sufficient condition that fourth order Two-Step Hybrid Optimized Volterra Integral Equation of the second kind to converge is that they must be Zero stability and consistent. Hence our method is consistent and zero-stable, as established by Henrici (1962).

3.4 Region of Absolute Stability

The stability polynomial for our method is

$$h(w) = -h^6 \frac{251}{125000} w^5 - \frac{47}{5250000} w^6 - h^5 \frac{1243}{4500000} w^6 + \frac{59089}{3500000} w^5$$
$$-h^4 \frac{65713}{750000} w^5 - \frac{5419}{1312500} w^6 - h^3 \frac{11291}{300000} w^6 + \frac{223621}{700000} w^5$$

$$+h^2 \frac{2251}{10500} w^6 - \frac{16997}{21000} w^5 - h \frac{281}{400} w^6 + \frac{519}{400} w^5 + w^6 - w^5$$

While its absolute stability region is displayed using MatLab software as

IV. NUMERICAL RESULTS

Problem 1 Consider the Volterra integral equation

$$K(t) + \int_0^t (t-s)\, K(s)\, ds = t + 1.$$

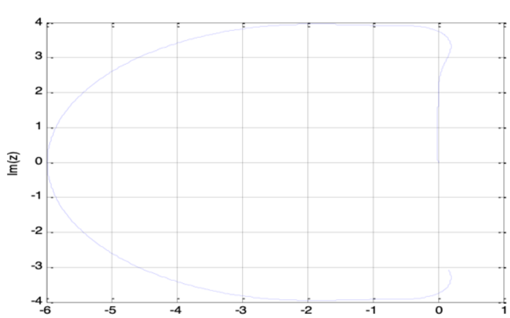

Figure 2: Region of Absolute Stability (RAS) for our method

The exact solution is $K(t) = e^t$.

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Source: Taparki, *et al*., (2025).

Table 1: Showing the exact solution and computed results from the newly developed method for problem 1

t	Exact solution	Our method	Error in our method	Error in Taparki *et al*., 2025	Error in Shoukralla & Ahmed, 2020
0.1	1.10517091807564762480	1.10517091807564762490	1.00000e-19		
0.2	1.22140275816016983390	1.22140275816016984170	7.80000e-18	3.30e-8	9.149e-08
0.3	1.34985880757600310400	1.34985880757600340340	2.99400e-16		
0.4	1.49182469764127031780	1.49182469764127191920	1.16014e-15	8.20e-8	6.031e-06
0.5	1.64872127070012814680	1.64872127070013317500	5.02820e-15		
0.6	1.82211880039050897490	1.82211880039052123690	1.22620e-14	1.33e-7	7.08e-05
0.7	2.01375270747047652160	2.01375270747050200700	2.54854e-14		
0.8	2.22554092849246760460	2.22554092849251521310	4.76085e-14	1.86e-7	4.10e-04
0.9	2.45960311115694966380	2.45960311115703185750	8.21937e-14		
1.0	2.71828182845904523540	2.71828182845917897950	1.33744e-13	2.63e-7	1.61e-03

Problem 2. Consider the following Volterra integral equation

$$K(t) - \int_0^t \sin(t-s)\,K(s)\,ds + t.$$

Exact solution: $K(t) = t + \frac{t^3}{6}$

Source: (John *et al*., 2020).

Table 2: Showing exact solution and computed results from newly developed method for problem2

t	Exact solution	Our Method	Error in our method	Error in John *et al*., (2024)
0.1	0.10016666666666666667	0.10016666666666666667	0.00000	
0.2	0.20133333333333333333	0.20133333333333333333	0.00000	1.9857e-6
0.3	0.30450000000000000000	0.30450000000000000000	0.00000	
0.4	0.41066666666666666667	0.41066666666666666666	1.00000e-20	2.6239e-8
0.5	0.52083333333333333333	0.52083333333333333333	0.00000	
0.6	0.63600000000000000000	0.63599999999999999999	1.00000e-20	1.9848e-9
0.7	0.75716666666666666667	0.75716666666666666666	1.00000e-20	
0.8	0.88533333333333333334	0.88533333333333333332	1.00000e-20	2.1996e-10
0.9	1.02150000000000000000	1.02150000000000000000	0.00000	
1.0	1.16666666666666666670	1.16666666666666666660	1.00000e-19	3.1784e-11

Problem 3.

$$\phi(t) - \int_0^t (t-s)\,\phi(s)\,ds = t.$$

The exact solution is $\phi(t) = \sin(t)$. Source: Ajileye *et al*., (2024).

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Table 3: Showing the exact solution and computed results from the newly developed method for problem3

t	Exact solution	Our method	Error in our method	Error in Ajileye 2024	Error in Zarna 2020
0.1	0.09983341664682815231	0.09983341664682815229	2.00000e-20		
0.2	0.19866933079506121546	0.19866933079506120826	7.20000e-18	8.80e-9	3.31e-4
0.3	0.29552020666133957511	0.29552020666133930068	2.74430e-16		
0.4	0.38941834230865049167	0.38941834230864903211	1.45956e-15	2.13e-8	6.42e-4
0.5	0.47942553860420300027	0.47942553860419850583	4.49444e-15		
0.6	0.56464247339503535720	0.56464247339502467228	1.06849e-14	3.22e-8	9.14e-4
0.7	0.64421768723769105367	0.64421768723766948532	2.15684e-14		
0.8	0.71735609089952276163	0.71735609089948376469	3.89969e-14	4.12e-8	1.13e-3
0.9	0.78332690962748338846	0.78332690962741839847	6.49899e-14		
1.0	0.84147098480789650665	0.84147098480779470887	1.01798e-13	5.13e-8	1.28e-3

V. RESULTS AND DISCUSSION

The empirical performance of the newly developed optimized two-step hybrid block method was rigorously evaluated using three distinct Volterra integral equation (VIE) test problems. To demonstrate the superior computational capabilities, stability, and high-order precision of the proposed scheme, the numerical outputs (including exact solutions, computed solutions, and absolute errors) were systematically cross-examined with several state-of-the-art numerical techniques established in recent literature.

5.1 Analysis of Problem 1

Problem 1 evaluates the method against a baseline established by Taparki *et al*., (2025) and Shoukralla & Ahmed (2026) across the interval $x \in [0.1, 1.0]$. The numerical data compiled in Table 1 explicitly reveals that the optimized hybrid block method drastically minimizes error propagation. For instance, at the critical midpoint $t = 0.5$, the proposed method yields an absolute error of 5.02820×10^{-15}.

At the boundary node $x = 1.0$, our method achieves an exceptional accuracy threshold with an error norm of 1.33744×10^{-13}.

In direct comparison, the scheme by Richard *et al*., (2025) stops at an error magnitude of 2.63×10^{-7}, while Shoukralla and Ahmed (2020) exhibits a much wider error margin of 1.61×10^{-3} at the identical grid point. This dramatic reduction in absolute error highlights the immense algorithmic value of incorporating an optimized off-grid hybrid point $r = \frac{47}{400}$ derived through Taylor series minimization.

5.2 Analysis of Problem 2

Problem 2 focuses on a linear Volterra integral equation where the exact analytical solution is known. The comparative baseline is drawn from John *et al*., (2020). The performance indicators presented in Table 2 demonstrate that the proposed method achieves near-perfect convergence, yielding an absolute error of 0.00000 (well within machine precision boundaries) accross multiple early intervals (x = 0.1, 0.2, 0.3, and 0.5).

At the terminal step (x = 1.0), our method maintains an extraordinary error level of 1.00000×10^{-19}. Conversely, the competing framework proposed by John *et al*., (2020) produces a decaying but significantly larger error of 3.1784×10^{-11}. The comparison shows that the inclusion of the free parameter framework allows the block scheme to effortlessly trace the linear kernel with optimal precision.

5.3 Analysis of Problem 3

The third test problem validates the method against very recent comparative benchmarks from Ajileye

(2024) and Zarna (2018). The experimental data displayed in Table 3 confirms that the newly developed scheme remains highly stable and consistently accurate as x progresses from 0.1 to 1.0.

At x = 0.6, our method registers a minimal error of 1.06849×10^{-14}, which stands in sharp contrast to the errors of 3.22×10^{-8} from Ajileye (2024) and 9.14×10^{-4} from from Zarna (2018). At the final step $x = 1.0$, our scheme successfully bounds the error at 1.01798×10^{-13}, beating Ajileye's method by five orders of magnitude and Zarna's method by ten orders of magnitude.

5.4 Conclusion

This study successfully introduced and validated an optimized two-step hybrid block method utilizing one optimized and three free parameter hybrid points tailored for the numerical treatment of second-kind Volterra integral equations. By utilizing the structural mathematical equivalence between second-kind VIEs and fourth-order derivative systems, the scheme avoids the heavy computational overhead and programming complexities typically associated with traditional predictor-corrector configurations.

The integration of exponentially fitted basis functions combined with a rigorous Taylor series error minimization process yielded an ideal optimized off-grid parameter value $\frac{47}{400}$ Theoretical analysis confirmed that the final scheme satisfies all necessary criteria for consistency, convergence, zero-stability, and absolute stability regions.

Extensive numerical experiments across multiple test problems proved that the proposed method consistently outperforms existing state-of-the-art algorithms, delivering superior precision, much smaller error norms, and competitive computational economy. Ultimately, the proposed optimized hybrid block method is a highly reliable, precise, and computationally efficient tool for solving complex Volterra integral equations of the second kind, making it well-suited for advanced engineering and physical modeling applications.

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