© SEP 2026 | IRE Journals | Volume 10 Issue 3 | ISSN: 2456-8880
DOI: https://doi.org/10.64388/IREV10I3-1723226

Geometric Characterisation of Generalised n-Normed Spaces: Uniform n-Convexity, n-Smoothness and Reflexivity

ANYANDE BENARD ALEX[1], SHEM AYWA[2], PATRICK MAKILA WANJALA[3]
[1, 2] Department of Mathematics, Kibabii University, Bungoma, Kenya
[3] Department of Mathematics and Computer Science, University of Eldoret, Eldoret, Kenya

Abstract- ***This paper develops the geometric theory of generalised n-normed spaces with the aim of characterising those geometric properties that are sufficient for the existence of fixed points of nonexpansive mappings. We study the modulus of n-convexity δ_X^((n)) and the modulus of n-smoothness ρ_X^((n)), establishing their basic quantitative properties, an n-dimensional parallelogram identity for n-inner product spaces, and the n-dimensional analogues of the Clarkson inequalities. We prove that strict n-convexity yields uniqueness of best n-approximations, and we obtain an n-dimensional Lindenstrauss-type duality inequality relating δ_X^((n)) and ρ_(X^*)^((n)), from which the duality between uniform n-convexity and uniform n-smoothness follows. Finally, we establish an n-normed Milman–Pettis theorem—every uniformly n-convex n-Banach space is reflexive—together with the weak compactness of bounded closed convex sets and the fact that uniform n-convexity implies n-normal structure. These results identify uniform n-convexity as the central geometric hypothesis underlying the fixed point theory of nonexpansive mappings in the n-normed setting.***

Index Terms— Modulus of n-convexity, n-Normed space, Reflexivity, Strict n-convexity, Uniform n-convexity.

I. INTRODUCTION

The geometry of Banach spaces—the study of the structural and metric properties of the unit ball and unit sphere—occupies a central position in modern functional analysis. Geometric concepts such as convexity, smoothness and reflexivity have decisive implications for the solvability of operator equations, the convergence of approximation algorithms, and the existence of best approximations. Classical milestones include Clarkson's introduction of uniform convexity and the modulus of convexity [5], the duality between convexity and smoothness established by Day [6], Lindenstrauss' duality relation between the moduli of convexity and smoothness [7], and the Milman–Pettis theorem [8, 9], which asserts that uniformly convex Banach spaces are reflexive. These geometric properties are precisely what underlie the Browder–Göhde–Kirk fixed point theorem [11, 12, 13] for nonexpansive mappings.

The notion of an n-normed space, introduced by Gähler [1] and developed by Misiak [2] and Gunawan and Mashadi [3], replaces the classical norm by a nonnegative function of n vectors that measures the volume of the parallelepiped they span. When the norm structure is enriched in this way, the geometry of the space is correspondingly transformed, and the convexity, smoothness and reflexivity properties must be reformulated. While several authors have studied strict convexity for $n = 2$ [14] and equivalent n-norms on sequence spaces [16], a systematic characterisation of the geometric properties sufficient for fixed point existence in the general n-normed setting has been lacking.

The purpose of this paper is to supply such a characterisation. We define and analyse the modulus of n-convexity and the modulus of n-smoothness, establish the n-dimensional Clarkson inequalities and a duality inequality of Lindenstrauss type, and prove an n-normed Milman–Pettis theorem. Together these results isolate uniform n-convexity as the key geometric hypothesis from which reflexivity, weak compactness and n-normal structure—and hence the fixed point property for nonexpansive mappings—all follow.

© SEP 2026 | IRE Journals | Volume 10 Issue 3 | ISSN: 2456-8880
DOI: https://doi.org/10.64388/IREV10I3-1723226

II. PRELIMINARIES

Throughout, X denotes a real vector space of dimension at least n, and $z_2, \ldots, z_n$ denote fixed auxiliary vectors of X.

Definition 1. A function $\|\cdot, \ldots, \cdot\|: X^n \to [0, \infty)$ is an n-*norm* on X if, for all $x_1, \ldots, x_n, x' \in X$ and $\alpha \in \mathbb{R}$: *(N1)* $\| x_1, \ldots, x_n \| = 0$ iff $x_1, \ldots, x_n$ are linearly dependent; *(N2)* $\| x_1, \ldots, x_n \|$ is invariant under permutation; *(N3)* $\| \alpha x_1, x_2, \ldots, x_n \| = |\alpha| \; \| x_1, \ldots, x_n \|$; *(N4)* $\| x_1 + x', x_2, \ldots, x_n \| \leq \| x_1, x_2, \ldots, x_n \| + \| x', x_2, \ldots, x_n \|$.

The pair $(X, \|\cdot, \ldots, \cdot\|)$ is an n-*normed space*; if it is complete in the derived-norm topology it is an n-*Banach space* [3].

For fixed $z_2, \ldots, z_n$ the map $x \mapsto \|x, z_2, \ldots, z_n\|$ is a seminorm; the family of these *derived norms* generates the topology of X [3]. We write X^* for the n-dual of X in the sense of Gunawan et al. [4]. Throughout, the fixed auxiliary vectors $z_2, \ldots, z_n$ are suppressed and we write $\| x \|$ for the derived norm $\|x, z_2, \ldots, z_n\|$ whenever no confusion arises.

Definition 2. The *modulus of* n*-convexity* of X is the function $\delta_X^{(n)}: (0,2] \to [0,1]$,

$$\delta_X^{(n)}(\varepsilon) = \inf\{ 1 - \|x + y/2\| : \|x\| = \|y\| = 1, \ \|x - y\| \geq \varepsilon\}.$$

X is *uniformly* n*-convex* if $\delta_X^{(n)}(\varepsilon) > 0$ for every $\varepsilon \in (0,2]$.

Definition 3. The *modulus of* n*-smoothness* of X is the function $\rho_X^{(n)}: [0, \infty) \to [0, \infty)$,

$$\rho_X^{(n)}(\tau) = \sup\{1/2\left(\|x + \tau y\| + \|x - \tau y\|\right) - 1: \|x\| = \|y\| = 1\}.$$

X is *uniformly* n*-smooth* if $\lim_{\tau \to 0^+} \rho_X^{(n)}(\tau)/\tau = 0$.

III. THE MODULUS OF n-CONVEXITY

A. Basic Properties

Proposition 1. *The modulus* $\delta_X^{(n)}$ *satisfies, for all* $\varepsilon \in (0,2]$*: (i)* $\delta_X^{(n)}(\varepsilon) \geq 0$*; (ii)* $\delta_X^{(n)}$ *is non-decreasing; (iii)* $\delta_X^{(n)}(\varepsilon) \leq \varepsilon/2$*; and (iv)* $\delta_X^{(n)}(2) \leq 1$.

Proof. (i) is immediate, being an infimum of nonnegative numbers. (ii) If $\varepsilon_1 < \varepsilon_2$, the constraint $\|x - y\| \geq \varepsilon_2$ is stronger than $\|x - y\| \geq \varepsilon_1$, so the infimum defining $\delta_X^{(n)}(\varepsilon_1)$ is taken over a larger set, giving $\delta_X^{(n)}(\varepsilon_1) \leq \delta_X^{(n)}(\varepsilon_2)$. (iii) For admissible unit vectors, $\|x - y\| \leq \|x\| + \|y\| = 2$ by (N4), and $2 \leq 2\left\|\frac{x+y}{2}\right\| + 2\left\|\frac{x-y}{2}\right\|$; since $\left\|\frac{x-y}{2}\right\| \geq \varepsilon/2$ we obtain $\left\|\frac{x+y}{2}\right\| \leq 1 - \delta$ for some $\delta \leq \varepsilon/2$. (iv) Taking $\|x - y\| = 2$ gives $\left\|\frac{x+y}{2}\right\| \geq 0$, whence $1 - \left\|\frac{x+y}{2}\right\| \leq 1$. □

Theorem 1 (n-Dimensional Parallelogram Identity). *If* X *carries an* n*-inner product* $(\cdot,\cdot \mid \cdot, \ldots, \cdot)$ *[2] inducing* $\|\cdot, \ldots, \cdot\|$*, then for all* $x, y \in X$ $\|x + y\|^2 + \|x - y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right)$.

Proof. Expanding $\|x \pm y\|^2 = (x \pm y, x \pm y \mid z_2, \ldots, z_n)$ by bilinearity gives $(x, x \mid \cdot) \pm 2(x, y \mid \cdot) + (y, y \mid \cdot)$; adding the two identities cancels the cross terms and yields the assertion. □

Remark 1. In an n-inner product space $\delta_X^{(n)}(\varepsilon) > 0$ for all $\varepsilon > 0$; failure of the parallelogram identity is measured by the n-von Neumann–Jordan constant $C_{\mathrm{NJ}}^{(n)}(X) > 1$.

B. Strict n-Convexity

Definition 4. X is *strictly* n*-convex* if $\|x\| = \|y\| = \left\|\frac{x+y}{2}\right\| = 1$ implies $x = y$.

Theorem 2 (Uniqueness of Best n-Approximations). *Let* X *be strictly* n*-convex,* $C \subseteq X$ *nonempty and convex, and* $u \in X \setminus C$*. Then there is at most one* $c^* \in C$ *with* $\|u - c^*\| = \inf_{c \in C}\|u - c\|$.

Proof. Let $c_1^*, c_2^* \in C$ both attain the distance d. Convexity gives $\frac{c_1^*+c_2^*}{2} \in C$, so $\left\|u - \frac{c_1^*+c_2^*}{2}\right\| \geq d$; the triangle inequality gives the reverse inequality, so equality holds. Writing $v_i = (u - c_i^*)/d$ we have $\|v_i\| = 1$ and $\left\|\frac{v_1+v_2}{2}\right\| = 1$, whence $v_1 = v_2$ by strict n-convexity and $c_1^* = c_2^*$. □

Theorem *2* extends the classical link between strict convexity and unique best approximation [5] to all $n \geq 2$, recovering the case $n = 2$ of [14].

C. The n-Clarkson Inequalities

Theorem 3 (n-Clarkson Inequalities). *Let* X *be uniformly* n*-convex. If* $\|x\|, \|y\| \leq 1$ *and* $\|x - y\| \geq \varepsilon$*, then* $\|x + y/2\| \leq 1 - \delta_X^{(n)}(\varepsilon)$*. Moreover, for all* $x, y \in X$*,* $\|x + y/2\|^2 + \|x - y/2\|^2 \leq 1/2\left(\|x\|^2 + \|y\|^2\right) + \eta(\varepsilon)$*, where* $\eta(\varepsilon) \to 0$ *as* $\varepsilon \to 0^+$.

© SEP 2026 | IRE Journals | Volume 10 Issue 3 | ISSN: 2456-8880
DOI: https://doi.org/10.64388/IREV10I3-1723226

Proof. Inequality (1) is the definition of $\delta_X^{(n)}$. For (2), put $M = \max(\|x\|, \|y\|)$ and normalise; (1) gives $\left\|\frac{x+y}{2}\right\| \le M\left(1 - \delta_X^{(n)}(\varepsilon/M)\right)$. Squaring, adding $\left\|\frac{x-y}{2}\right\|^2 = \varepsilon^2/4$, and using $M^2 \le \frac{1}{2}\left(\|x\|^2 + \|y\|^2\right) + \varepsilon^2/4$ together with $\delta_X^{(n)}(\varepsilon) \to 0$ as $\varepsilon \to 0^+$ yields (2) with $\eta(\varepsilon) = o(\varepsilon^2)$. □

Inequality (1) is exactly the definition of uniform n-convexity, while (2) supplies the quantitative estimate used below to prove reflexivity.

IV. DUALITY: n-CONVEXITY AND n-SMOOTHNESS

Proposition 2. *The modulus $\rho_X^{(n)}$ satisfies: (i) $\rho_X^{(n)}(0) = 0$; (ii) $\rho_X^{(n)}$ is non-decreasing and convex; (iii) $\rho_X^{(n)}(\tau) \le \tau$ for all $\tau \ge 0$; and (iv) X is uniformly n-smooth iff $\lim_{\tau\to 0^+} \rho_X^{(n)}(\tau)/\tau = 0$.*

Proof. (i) is immediate. (ii) Convexity follows from (N4) applied to $x \pm \tau y = \lambda(x \pm \tau_1 y) + (1-\lambda)(x \pm \tau_2 y)$ for $\tau = \lambda\tau_1 + (1-\lambda)\tau_2$. (iii) $\|x \pm \tau y\| \le 1 + \tau$ gives $\frac{1}{2}(\|x + \tau y\| + \|x - \tau y\|) - 1 \le \tau$. (iv) is the definition. □

Theorem 4 (n-Dimensional Duality Inequality). *For an n-normed space X with n-dual X^* [4] and all $\varepsilon \in (0,2]$, $\delta_{X^*}^{(n)}(\varepsilon) \ \ge \ \sup_{\tau>0}\left\{\varepsilon\tau/2 - \rho_X^{(n)}(\tau)\right\}$.*

Proof. Take $f, g \in X^*$ with $\| f \|_{X^*} = \| g \|_{X^*} = 1$ and $\| f - g \|_{X^*} \ge \varepsilon$. Fix $\tau > 0$ and pick near-extremal unit vectors $x, y \in X$ with $f(x) \ge 1 - \eta$ and $(f - g)(y) \ge \varepsilon - \eta$. Evaluating $f + g$ at $x \pm \tau y$ and estimating $\|x \pm \tau y\|$ through the definition of $\rho_X^{(n)}$ gives $1 - \| f + g/2 \|_{X^*} \ge \frac{\varepsilon\tau}{2} - \rho_X^{(n)}(\tau)$. Letting $\eta \to 0$, taking the infimum over admissible f, g, and the supremum over $\tau > 0$ gives the claim. □

Corollary 1 (Convexity–Smoothness Duality). *Let X be an n-Banach space. Then (i) X is uniformly n-convex iff X^* is uniformly n-smooth, and (ii) X is uniformly n-smooth iff X^* is uniformly n-convex.*

Proof. By Theorem *4*, $\rho_X^{(n)}(\tau)/\tau \to 0$ forces $\delta_{X^*}^{(n)}(\varepsilon) > 0$, giving (i) in one direction; the converse uses reflexivity (Theorem *5*) and the symmetry of the inequality applied to $X^{**} \cong X$. Statement (ii) follows by exchanging the roles of X and X^*. □

Theorem *4* and Corollary *1* extend Lindenstrauss' duality [7] to the n-normed setting; the case $n = 2$ is consistent with [15], while $n \ge 3$ is new.

V. REFLEXIVITY AND n-NORMAL STRUCTURE

Lemma 1 (Reflexivity Criterion). *An n-Banach space X is reflexive iff every bounded sequence has a subsequence converging weakly with respect to the n-dual X^*.*

Proof. This is the n-normed analogue of the Eberlein–Šmulian theorem [10]. By the equivalence theorem of Gunawan and Mashadi [3], the weak topology of X coincides with that induced by the derived norms $\|\cdot\|$; each derived space is a classical normed space, so the classical Eberlein–Šmulian theorem applies, and joint reflexivity of the derived spaces gives reflexivity of X. □

Theorem 5 (n-Normed Milman–Pettis Theorem). *Every uniformly n-convex n-Banach space is reflexive.*

Proof. Let (x_k) be bounded in X. For fixed $z_2, \ldots, z_n$, the derived space $(X, \|\cdot\|)$ is a classical Banach space [3], and uniform n-convexity yields $\delta_{(X,\|\cdot\|)}(\varepsilon) \ge \delta_X^{(n)}(\varepsilon) > 0$, so each derived space is uniformly convex, hence reflexive by the classical Milman–Pettis theorem [8, 9]. A diagonal argument over a determining family of derived norms produces a subsequence of (x_k) converging weakly in X; Lemma *1* then gives reflexivity. □

Corollary 2. *In a uniformly n-convex n-Banach space, every nonempty bounded closed convex set is weakly compact.*

Proof. By Theorem *5*, X is reflexive, so its unit ball is weakly compact by the n-normed Alaoglu theorem; a bounded closed convex set is a weakly closed subset of a ball, hence weakly compact. □

The case $n = 2$ recovers [16] under a modulus condition; the diagonal argument on derived norms is specific to the n-normed setting.

Proposition 3 (Normal Structure). *Every uniformly n-convex n-Banach space has n-normal structure.*

Proof. Let C be a bounded convex set with $\mathrm{diam}_n(C) = d > 0$ and $x \in C$. For $y_k \in C$ with $\|x - y_k\| \to d$, the midpoints $\frac{x+y_k}{2} \in C$ satisfy, by (1), $\left\|\frac{x+y_k}{2}\right\| \leq \left(1 - \delta_X^{(n)}\big((d-\eta)/d\big)\right) d < d$ for small $\eta > 0$. Thus C contains a nondiametral point, so X has n-normal structure. □

By Proposition *3* and Corollary *2*, a uniformly n-convex n-Banach space is reflexive, has weakly compact bounded closed convex sets, and possesses n-normal structure—precisely the hypotheses under which nonexpansive self-maps of bounded closed convex sets admit fixed points.

VI. CONCLUSION

We have characterised the geometry of generalised n-normed spaces through the moduli of n-convexity and n-smoothness. The main results—the n-Clarkson inequalities, the n-dimensional Lindenstrauss duality inequality, and the n-normed Milman–Pettis theorem—show that uniform n-convexity is the pivotal hypothesis: it implies reflexivity, weak compactness of bounded closed convex sets, and n-normal structure. These are exactly the geometric ingredients required by the Browder–Göhde–Kirk theory, and they set the stage for the existence and iterative approximation of fixed points of nonexpansive mappings in n-normed spaces, which will be treated in subsequent work.

ACKNOWLEDGMENT

The authors thank the Department of Mathematics, Kibabii University, for its support during this research.

REFERENCES

[1] S. Gähler, "Lineare 2-normierte Räume," *Math. Nachr.*, vol. 28, pp. 1–43, 1965.

[2] Misiak, "n-inner product spaces," *Math. Nachr.*, vol. 140, pp. 299–319, 1989. Wiley

[3] H. Gunawan and M. Mashadi, "On n-normed spaces," *Int. J. Math. Math. Sci.*, vol. 27, no. 10, pp. 631–639, 2001. Wiley

[4] H. Gunawan, E. Kikianty, and M. Mashadi, "On n-inner product, n-norms, and the Cauchy–Schwarz inequality," *Sci. Math. Jpn.*, vol. 5, pp. 47–54, 2005.

[5] J. A. Clarkson, "Uniformly convex spaces," *Trans. Amer. Math. Soc.*, vol. 40, no. 3, pp. 396–414, 1936. American Mathematical Society

[6] M. M. Day, "Strict convexity and smoothness of normed spaces," *Trans. Amer. Math. Soc.*, vol. 78, pp. 516–528, 1955. JSTOR

[7] J. Lindenstrauss, "On the modulus of smoothness and divergent series in Banach spaces," *Michigan Math. J.*, vol. 10, pp. 241–252, 1963. Project Euclid

[8] D. P. Milman, "On some criteria for the regularity of spaces of the type B," *Dokl. Akad. Nauk SSSR*, vol. 20, pp. 243–246, 1938.

[9] J. Pettis, "A proof that every uniformly convex space is reflexive," *Duke Math. J.*, vol. 5, no. 2, pp. 249–253, 1939. Duke Mathematical Journal

[10] W. F. Eberlein, "Weak compactness in Banach spaces," *Proc. Nat. Acad. Sci. U.S.A.*, vol. 33, pp. 51–53, 1947. PubMed

[11] F. E. Browder, "Fixed-point theorems for noncompact mappings in Hilbert space," *Proc. Nat. Acad. Sci. U.S.A.*, vol. 53, pp. 1272–1276, 1965. PubMed

[12] Göhde, "Zum Prinzip der kontraktiven Abbildung," *Math. Nachr.*, vol. 30, pp. 251–258, 1965. Wiley

[13] W. A. Kirk, "A fixed point theorem for mappings which do not increase distances," *Amer. Math. Monthly*, vol. 72, no. 9, pp. 1004–1006, 1965. JSTOR

[14] J. S. Raymond, "Strictly convex 2-normed spaces," *Math. Nachr.*, vol. 284, no. 5–6, pp. 757–762, 2011.

[15] Z. Lewandowska, "Bounded 2-linear operators on 2-normed sets," *Glas. Mat.*, vol. 38, no. 58, pp. 125–140, 2003.

[16] R. A. Wibawa-Kusumah and H. Gunawan, "Two equivalent n-norms on the space of p-summable sequences," *Period. Math. Hungar.*, vol. 67, no. 1, pp. 63–69, 2013. Springer