© SEP 2026 | IRE Journals | Volume 10 Issue 3 | ISSN: 2456-8880
DOI: https://doi.org/10.64388/IREV10I3-1723311

Vague Differential Equations on Smooth Manifolds: A Foundational Framework for Vague Vector Fields, Existence, Uniqueness and Geometric Dynamics

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Abstract- ***Differential equations on manifolds provide a natural language for dynamics when the state space is curved, constrained, or intrinsically geometric. In many applications, however, the available information is incomplete, imprecise, or expressed in terms of degrees of truth and falsity rather than a single numerical value. This paper develops a rigorous framework for vague differential equations on smooth manifolds. A vague quantity is represented by a pair (t,f), where t is a truth-membership degree, f is a false-membership degree, and $0 \leq t+f \leq 1$; the associated admissible interval is [t,1−f]. We distinguish carefully between the ordinary manifold state and vague coefficients, and then define vague scalar fields, vague tangent vectors, vague vector fields, vague differential equations, and coordinate representations. The central construction is a pair-valued differential equation whose truth and falsity components evolve according to differentiable vector-field equations, while the underlying state remains on the manifold. We establish basic invariance, coordinate covariance, local existence and uniqueness under Lipschitz hypotheses, reduction to classical differential equations, and a Grönwall-type stability estimate for the component fields. Explicit examples are given on the circle, sphere, and Euclidean charts, including a vague rotational flow on S^1 and a vague gradient-type flow on S^2. A numerical Euler scheme in local coordinates is also formulated, together with an admissibility condition ensuring that vague degrees remain in the unit square. The paper emphasizes that a mathematically sound vague differential-equation theory must specify its notion of derivative and must not confuse vague membership with an ordinary real-valued state. The framework is intended as a starting point for further work on vague differential geometry, vague dynamical systems, optimization, control, and uncertain geometric computation.***

Keywords- Vague set; vague differential equation; smooth manifold; tangent bundle; vague vector field; differential geometry; existence and uniqueness; geometric dynamics; uncertain dynamical system; vague membership.

I. INTRODUCTION

Differential equations are traditionally formulated on Euclidean spaces. An equation $x'(t)=F(t,x(t))$ assumes that the state x(t) belongs to a vector space in which subtraction, differentiation, and linear combinations are directly available. Many important systems do not have this global Euclidean structure. The attitude of a rigid body, the position of a particle constrained to a surface, the state of a mechanical system with holonomic constraints, and numerous geometric optimization problems naturally evolve on manifolds. A smooth manifold M is locally modeled on R^n, but its global topology and geometry may be non-Euclidean. Differential equations on manifolds therefore require tangent spaces, vector fields, charts, and, for second-order equations, connections or metrics. The resulting theory is well established in differential geometry and dynamical systems.

A different issue arises when the model itself contains incomplete or imprecise information. A conventional fuzzy set assigns one membership value to an element. A vague set, in the terminology of Gau and Buehrer, separates evidence for truth from evidence for falsity. For an element x, a vague value may be represented as $[t(x),1-f(x)]$, subject to $t(x)+f(x)\leq 1$. The interval records the unresolved part of the information.

The purpose of this paper is not to replace manifold differential equations with an informal interval notation. Instead, we construct a precise object: a vague scalar or vector field is represented by differentiable truth and falsity component fields satisfying an admissibility constraint. A vague differential equation is then a system of differential equations for those components, coupled to an ordinary manifold trajectory when appropriate. This

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DOI: https://doi.org/10.64388/IREV10I3-1723311

separation is important because a tangent vector belongs to T_pM, whereas a vague membership pair belongs to a coefficient space.

The principal contributions are: (i) formal definitions of vague scalar fields and vague vector fields on a smooth manifold; (ii) a coordinate-invariant formulation of vague first-order differential equations; (iii) existence, uniqueness, invariance, and stability results under standard hypotheses; (iv) explicit manifold examples; and (v) a practical local-coordinate numerical scheme. The framework also identifies limitations and open problems, including intrinsic interval-valued tangent dynamics and second-order vague equations.

II. MATHEMATICAL PRELIMINARIES

2.1 Smooth manifolds

Let M be a smooth n-dimensional manifold. A chart is a pair (U,φ), where U is open in M and $\varphi:U\to\varphi(U)\subset\mathbb{R}^n$ is a homeomorphism with the required smooth compatibility with other charts. The tangent space at $p\in M$ is denoted T_pM and the tangent bundle is $TM=\cup_{p\in M}T_pM$.

A smooth vector field X on M assigns to every $p\in M$ a tangent vector $X(p)\in T_pM$ such that its coordinate components are smooth. If $\gamma:I\to M$ is a differentiable curve, then $\gamma'(t)\in T_{\gamma(t)}M$.

2.2 Classical differential equations on manifolds

A first-order autonomous differential equation on M is written

$$\gamma'(t) = X(\gamma(t)),\ \gamma(t_0) = p_0,$$

where X is a smooth vector field. In a local chart $x=\varphi\circ\gamma$, the equation becomes $x'(t)=F(x(t))$, where F is the coordinate representation of X. Standard local existence and uniqueness follow from the corresponding Euclidean theorem.

For a Riemannian manifold (M,g), a geodesic γ satisfies $\nabla_{\gamma'}\gamma'=0$. In coordinates this is $\ddot{\gamma}^k+\Gamma^k_{ij}\dot{\gamma}^i\dot{\gamma}^j=0$. Thus geometry enters differential equations through the tangent bundle, connection, metric, or other geometric structures.

2.3 Vague values

A vague value is modeled by an ordered pair $v=(t,f)$ satisfying

$$0 \le t \le 1,\quad 0 \le f \le 1,\quad t+f \le 1.$$

The associated vague interval is

$$I(v) = [t,\ 1-f].$$

Here t represents evidence supporting a proposition and f represents evidence supporting its negation. The quantity $1-t-f$ measures the residual or indeterminate information. The representation is exact only after the semantic interpretation of t and f has been fixed.

Let V denote the set of all admissible vague values. Classical values embed into V through $c\mapsto(c,1-c)$, for $c\in[0,1]$. Hence a classical membership value is a special case with no residual interval width.

III. VAGUE SCALAR FIELDS ON MANIFOLDS

3.1 Definition

Definition 3.1. Let M be a smooth manifold. A differentiable vague scalar field on M is a map

$$A:M\to V,\quad A(p)=(T(p),F(p)),$$

where $T,F:M\to[0,1]$ are differentiable and $T(p)+F(p)\le1$ for every $p\in M$. Its interval realization is the set-valued map

$$\bar{A}(p)=[T(p),1-F(p)].$$

The lower endpoint T is the truth-membership component, the upper endpoint $1-F$ is the complement of the falsity component, and the width

$$w_A(p)=1-F(p)-T(p)$$

measures residual vagueness.

3.2 Differential of a vague scalar field

Definition 3.2. For $A=(T,F)$, define its differential at p in the componentwise sense by

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DOI: https://doi.org/10.64388/IREV10I3-1723311

$dA_p(v) = (dT_p(v), dF_p(v)), \quad v \in T_pM.$

This is not itself a single real covector. Rather, it is a pair of covectors (dT_p,dF_p). Consequently the natural differential object is a pair-valued cotangent field.

Proposition 3.3. If T and F are C^k, then dT and dF are C^{k-1}. If $T+F\leq 1$ on M, the vague field is admissible independently of whether its differential is zero or nonzero.

Proof. The first statement follows from ordinary differential calculus. The second is simply the pointwise admissibility condition; differentiation describes local variation and does not alter the definition of admissibility. ■

IV. VAGUE TANGENT AND VECTOR FIELDS

4.1 Motivation

A common source of ambiguity is the expression 'a vague tangent vector'. A tangent vector at p is an element of the ordinary vector space T_pM. A vague tangent vector should therefore be defined as structured information about tangent vectors, rather than by simply multiplying a tangent vector by a vague number without specifying the operation.

For the foundational theory in this paper, we use a pair-of-vector-fields representation.

4.2 Definition

Definition 4.1. A differentiable vague vector field on M is a pair

$\mathcal{X}=(X_T,X_F),$

where X_T and X_F are differentiable vector fields on M, together with an admissibility interpretation specified by a scalar vague weight A=(T,F). The effective interval-weighted field is represented formally by

$\mathcal{I}(\mathcal{X},A)(p) = \{ \lambda X_T(p) + \mu X_F(p) : \lambda\in[T(p),1-F(p)], \mu\in[T(p),1-F(p)] \},$

when the application requires a set of possible tangent vectors. This construction is deliberately separated from the basic vague ODE, because interval arithmetic in tangent spaces requires a choice of local frame or a bundle-level set-valued formalism.

For the core existence theory below, we therefore define a vague trajectory through component equations rather than claiming that an interval is itself a tangent vector.

V. DEFINITION OF VAGUE DIFFERENTIAL EQUATIONS ON MANIFOLDS

5.1 Vague coefficient equation with manifold state

Definition 5.1. Let M be a smooth manifold and V the vague-value space. A first-order vague differential equation consists of

$\gamma'(t)=X(\gamma(t),t;T(t),F(t)),$
$T'(t)=a(t,\gamma(t),T(t),F(t)),$
$F'(t)=b(t,\gamma(t),T(t),F(t)),$

subject to $\gamma(t)\in M$ and $0\leq T(t),F(t)$, $T(t)+F(t)\leq 1$. Here X is a time-dependent vector field and a,b are scalar evolution laws.

This definition distinguishes the geometric state γ from the vague information variables T and F. It is especially suitable when uncertainty describes confidence in a model coefficient, sensor classification, material state, or control rule.

5.2 Vague vector-field flow

A simpler autonomous model is obtained when a vague parameter determines the vector field:

$\gamma'(t)=X_T(\gamma(t))\, T(t) + X_F(\gamma(t))\, [1-F(t)],$

in a local trivialization or in applications where the scalar coefficients have an agreed geometric meaning. A coordinate-free version is obtained by defining a smooth bundle map from the vague-value space into vector fields. The manuscript therefore treats such weighted expressions as model-dependent rather than universally canonical.

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DOI: https://doi.org/10.64388/IREV10I3-1723311

5.3 Reduction to classical equations
If $T=c$ and $F=1-c$ are constants, then the vague interval collapses to $\{c\}$. If the model is constructed so that X depends only on this classical value, the vague differential equation reduces to an ordinary manifold differential equation. This reduction property is essential for consistency.

VI. COORDINATE REPRESENTATION AND COVARIANCE

Let (U,φ) be a chart on M and write $x=\varphi(\gamma)$. Suppose $\gamma'(t)=X(t,\gamma,T,F)$.

Then there is a coordinate function G such that $x'(t)=G(t,x(t),T(t),F(t))$.

If $y=\psi\circ\varphi^{-1}(x)$ is another coordinate system, then $y'(t)=D(\psi\circ\varphi^{-1})(x(t))\, G(t,x(t),T(t),F(t))$.

The vague variables T and F are scalar quantities and therefore do not acquire the Jacobian factor. This gives the correct transformation law: the manifold velocity transforms as a tangent vector, while vague scalar information transforms as scalar data.

Theorem 6.1 (coordinate consistency). If the local coordinate representation satisfies the classical tangent transformation law above, then a solution described in one chart corresponds to the same geometric trajectory described in any overlapping chart.

Proof. Let $y=h(x)$, where $h=\psi\circ\varphi^{-1}$. By the chain rule, $y'=Dh(x)x'$. Substituting $x'=G$ gives $y'=Dh(x)G$. Thus both coordinate descriptions represent the same tangent vector $\gamma'(t)$. Since T and F are scalar fields along the trajectory, their values are unchanged by the coordinate transition. ■

VII. EXISTENCE AND UNIQUENESS

7.1 Local theorem
Consider the augmented state space $M\times D$, where

$$D=\{(T,F)\in\mathbb{R}^2: T\geq 0,\ F\geq 0,\ T+F\leq 1\}.$$

Let
$\mathcal{F}(t,p,T,F)=(X(t,p,T,F),\ a(t,p,T,F),\ b(t,p,T,F))$.

In a local chart, the system becomes an ordinary Euclidean ODE in R^{n+2}.

Theorem 7.1. Let X,a,b be continuous in t and locally Lipschitz in the state variables in a coordinate neighborhood of (t_0,p_0,T_0,F_0). Then there exists $\varepsilon>0$ and a unique local solution

$$(\gamma(t),T(t),F(t)),\quad |t-t_0|<\varepsilon,$$

satisfying $\gamma(t_0)=p_0$, $T(t_0)=T_0$, $F(t_0)=F_0$, provided the solution remains in the chosen chart.

Proof. Choose a chart $\varphi:U\to R^n$ around p_0. The coordinate system is $z=(x,T,F)\in R^{n+2}$. The transformed right-hand side is continuous in t and locally Lipschitz in z by hypothesis and smoothness of the chart transformation. The Picard–Lindelöf theorem gives a unique local Euclidean solution. Mapping $x(t)$ back by φ^{-1} gives $\gamma(t)$. ■

7.2 Global continuation
If M is compact and the vector field and vague component functions are globally Lipschitz in suitable bounded local representations, standard continuation arguments can extend the solution until any prescribed finite time, provided the admissibility region D is invariant. Compactness removes escape of the manifold component to infinity, while invariance prevents the vague variables from leaving their permitted domain.

VIII. ADMISSIBILITY AND INVARIANCE OF VAGUE VALUES

Existence of T and F as real-valued solutions does not automatically guarantee that they remain vague values. A separate invariance condition is required.

8.1 Tangency conditions on the triangular domain
Recall

$$D=\{(T,F): T\geq 0,\ F\geq 0,\ T+F\leq 1\}.$$

A sufficient boundary condition for invariance is:
- on $T=0$, require $a(t,p,0,F)\geq 0$;
- on $F=0$, require $b(t,p,T,0)\geq 0$;
- on $T+F=1$, require $a(t,p,T,F)+b(t,p,T,F)\leq 0$.

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DOI: https://doi.org/10.64388/IREV10I3-1723311

Theorem 8.1. Assume the right-hand side is locally Lipschitz and satisfies the three boundary inequalities above for all relevant states. Then D is positively invariant: an initial condition in D produces $T(t),F(t)\in D$ for every time for which the solution exists.

Proof. The vector field points inward or tangent on every boundary face of the closed convex triangle D. Suppose a solution starting in D first exits D. At the first exit time it must touch a boundary face while its derivative points outward. This contradicts the corresponding boundary inequality. Hence no first exit occurs. ■

This theorem is a basic consistency requirement for a vague differential equation. A numerical method should also be designed or corrected so that its computed pair remains in D.

IX. STABILITY OF VAGUE SOLUTIONS

Let two solutions of the same local coordinate system be $z_1=(x_1,T_1,F_1)$ and $z_2=(x_2,T_2,F_2)$. Suppose the right-hand side is Lipschitz with constant L on a common domain. Then

$$\|z_1(t)-z_2(t)\| \leq \|z_1(t_0)-z_2(t_0)\|\, e^{L|t-t_0|}.$$

Theorem 9.1. Under the preceding Lipschitz assumptions, the solution map is locally stable with respect to initial manifold coordinates and vague information.

Proof. The difference of the two solutions satisfies
$\|z_1(t)-z_2(t)\|\leq\|z_1(t_0)-z_2(t_0)\|+L\int\|z_1(s)-z_2(s)\|ds$.

Grönwall's inequality gives the result. ■

For an intrinsic statement, distances may be measured using a Riemannian metric on a compact neighborhood. Local equivalence of coordinate norms then converts the estimate into a geometric local stability estimate.

X. EXAMPLE I: A VAGUE ROTATIONAL EQUATION ON THE CIRCLE

Let $M=S^1$ and use the standard embedding $\gamma(t)=(x(t),y(t))$ with $x^2+y^2=1$. The rotational vector field is

$$X(x,y)=(-y,x).$$

Consider the vague scalar variables T,F satisfying

$$T'=\alpha(1-T-F),$$
$$F'=\beta(1-T-F),$$

where $\alpha,\beta\geq 0$ and $T_0+F_0\leq 1$. The manifold state is governed by

$$\gamma'(t)=c(t)X(\gamma(t)),\quad c(t)=T(t)+[1-F(t)]/2.$$

Since γ' is always tangent to S^1, the circle is invariant. Indeed

$$d/dt(x^2+y^2)=2x(-cy)+2y(cx)=0.$$

The vague width is $w=1-T-F$, and
$w'=-(\alpha+\beta)w$.

Hence

$$w(t)=w_0e^{-(\alpha+\beta)(t-t_0)}.$$

Thus the residual vagueness decreases exponentially when $\alpha+\beta>0$. The angular speed $c(t)$ is correspondingly driven by the evolving vague information.

This example illustrates an important modeling distinction. The trajectory itself is an ordinary point of S^1, while the speed law is vague. Consequently the geometric constraint is exact even though the information determining the speed is incomplete.

XI. EXAMPLE II: VAGUE DYNAMICS ON THE SPHERE

Let $M=S^2$. Consider a rotation about the z-axis:
$X(x,y,z)=(-y,x,0)$.

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DOI: https://doi.org/10.64388/IREV10I3-1723311

The classical equation $\gamma'=cX$ gives motion along latitude circles. Introduce vague coefficient variables T,F with

$$T'=\alpha(1-T-F),\quad F'=\beta(1-T-F),$$

and define

$$\gamma'=c(T,F)X(\gamma),\quad c(T,F)=T+q(1-F),$$

where $q\in[0,1]$. Since X is tangent to S^2, the sphere is invariant:

$$d/dt(x^2+y^2+z^2)=2c[-xy+xy+0]=0.$$

The vague dynamics changes the angular velocity while preserving the manifold constraint. If $q=1/2$, the coefficient depends symmetrically on the lower and upper endpoints of the vague interval.

XII. EXAMPLE III: VAGUE GRADIENT-TYPE FLOW

Let (M,g) be a Riemannian manifold and let $E:M\to\mathbb{R}$ be a smooth energy function. The negative gradient field $-\mathrm{grad}_g E$ is tangent to M. A vague modulation of the descent rate can be defined by

$$\gamma'(t)=-\rho(T(t),F(t))\ \mathrm{grad}_g E(\gamma(t)),$$

where $\rho:D\to[0,\infty)$ is smooth. Along a solution,

$$dE(\gamma(t))/dt=-\rho(T,F)\ \|\mathrm{grad}_g E\|^2_g \leq 0.$$

Therefore, whenever $\rho\geq 0$, the energy is nonincreasing even though the descent rate is controlled by vague information.

If ρ vanishes at some admissible vague states, the system may stall. If ρ is bounded below by a positive constant on the relevant region, the energy decreases strictly away from critical points. This provides a direct connection with geometric optimization under imprecise information.

XIII. VAGUE GEODESIC-TYPE EQUATIONS

A second-order equation requires additional care because $\ddot{\gamma}(t)$ is not intrinsically a tangent vector on a general manifold. A connection ∇ is needed. A vague acceleration law may therefore be written

$$\nabla_{\gamma'}\gamma' = G_T(t,\gamma,\gamma')T(t)+G_F(t,\gamma,\gamma')F(t),$$

where G_T and G_F are tangent-vector-valued fields along the curve. The vague variables satisfy their own evolution equations.

In a coordinate chart,

$$\ddot{\gamma}^k+\Gamma^k_{ij}\dot{\gamma}^i\dot{\gamma}^j = G_T^k\, T+G_F^k\, F.$$

This is a legitimate manifold equation because the left-hand side is the covariant acceleration. A paper claiming a vague second derivative on a manifold without specifying a connection would be incomplete. The connection-based formulation avoids that ambiguity.

XIV. NUMERICAL APPROXIMATION

14.1 Local Euler scheme

In a chart, write $z=(x,T,F)$ and $z'=\mathcal{F}(t,z)$. A basic Euler approximation is

$$z_{n+1}=z_n+h\mathcal{F}(t_n,z_n).$$

After each step, one must ensure that x_n remains inside the coordinate domain. More importantly, the computed (T_n,F_n) may leave D even when the continuous system is invariant.

14.2 Projection onto the vague domain

A practical correction is to replace

(T_{n+1},F_{n+1}) by its Euclidean projection onto $D=\{(T,F):T\geq 0,F\geq 0,T+F\leq 1\}$.

Projection is not part of the continuous differential equation and therefore changes the numerical method. It should be reported explicitly in computational work. For high-accuracy calculations, structure-preserving or adaptive methods may be preferable.

14.3 Local chart switching

For a manifold such as S^2, a single chart cannot cover the whole space. A numerical implementation must switch charts before leaving a chart domain, or use an embedded/geometric integrator that preserves the manifold constraint. The vague variables are unaffected by chart changes because they are scalar quantities.

XV. A GENERAL VAGUE MANIFOLD DYNAMICAL SYSTEM

The previous constructions can be unified on the product space $M\times D$. Let

$$\mathcal{F}(t,p,v)=(X(t,p,v),\ H(t,p,v)),\quad v=(T,F)\in D,$$

where X is tangent to M and H is tangent to D in the invariance sense. The vague manifold system is

$$p'=X(t,p,v),\quad v'=H(t,p,v).$$

This formulation is mathematically equivalent locally to an ordinary differential equation on the manifold-with-boundary $M\times D$. The adjective 'vague' identifies the interpretation of the auxiliary variables, not a relaxation of the differential-equation axioms.

Theorem 15.1. If X is a smooth time-dependent vector field in p and locally Lipschitz in v, while H is locally Lipschitz in (p,v) and satisfies the boundary conditions of Theorem 8.1, then the augmented system has a unique local solution and D is positively invariant.

Proof. This is the combination of Theorem 7.1 and Theorem 8.1 applied to the augmented state space $M\times D$. ■

XVI. STRUCTURAL PROPERTIES

16.1 Classical limit

A vague theory should possess a classical limit. If $T=c$ and $F=1-c$, the residual width is zero. Any model function depending continuously on (T,F) therefore reduces to its classical parameterized counterpart.

16.2 Zero-width versus nonzero-width information

The quantity $w=1-T-F$ separates precise and imprecise information. When $w=0$, the vague interval collapses. When $w>0$, the model contains unresolved information. The differential equation may either reduce w over time, preserve it, or increase it, subject to admissibility.

16.3 Invariant submanifolds

Suppose $N\subset M$ is an embedded submanifold and $X(t,p,T,F)\in T\_pN$ whenever $p\in N$. Then a solution starting in N remains in N, by uniqueness. This is useful for constrained mechanical systems and geometric computation.

16.4 Equilibria

A vague equilibrium is a state (p^*,T^*,F^*) satisfying

$$X(t,p^*,T^*,F^*)=0,\quad a(t,p^*,T^*,F^*)=0,\quad b(t,p^*,T^*,F^*)=0$$

in an autonomous formulation, or the corresponding invariant condition in a time-dependent system. Thus equilibria can arise from geometry, vague information, or both.

XVII. COMPARISON WITH RELATED UNCERTAINTY MODELS

Vague differential equations should be distinguished from several neighboring approaches.

- Fuzzy differential equations usually use fuzzy-valued states or coefficients and require a chosen fuzzy derivative, such as a Hukuhara-type or generalized derivative.
- Interval differential equations propagate set-valued uncertainty and commonly use differential inclusions or interval arithmetic.
- Stochastic differential equations model randomness through probability and stochastic integration.
- Vague differential equations, as defined here, represent truth/falsity evidence through the pair (T,F), with the associated interval $[T,1-F]$.

These models are not interchangeable. A rigorous study must specify what is uncertain, which derivative is used, and whether the state itself or only its coefficients are vague.

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DOI: https://doi.org/10.64388/IREV10I3-1723311

XVIII. APPLICATIONS

18.1 Geometric control
A controller on a manifold may receive a confidence level about the validity of a control action. The pair (T,F) can evolve from sensor evidence, while the state remains on the configuration manifold. This is relevant to attitude dynamics and constrained control.

18.2 Robotics
Robot configurations often belong to manifolds such as SO(2), SO(3), or products of Lie groups. Vague variables can encode confidence in perception, contact classification, or task completion. A geometric differential equation then propagates the robot state without abandoning the configuration constraint.

18.3 Medical and biological modeling
Shape spaces, orientation spaces, and other geometric state spaces arise in imaging and biological dynamics. Vague evidence can be associated with a classification or model-selection layer while the underlying state evolves geometrically.

18.4 Optimization
The vague gradient flow of Section 12 provides a simple model of optimization where the descent rate is modulated by incomplete information. More sophisticated versions could allow vague objective functions, constraints, or adaptive metrics.

18.5 Physics and geometric mechanics
Many physical systems evolve on manifolds defined by constraints or symmetry. Vague parameters can represent incomplete model information. The main mathematical requirement is to preserve the geometric structure separately from the uncertainty representation.

XIX. COMPUTATIONAL RESEARCH PROGRAM

A practical research program based on this framework can proceed in stages:

1. Construct benchmark vague ODEs on S^1, S^2, tori, and Lie groups.
2. Compare Euler, Runge–Kutta, and manifold-aware integrators.
3. Measure residual vagueness $w(t)=1-T(t)-F(t)$ and its relationship with trajectory error.
4. Develop differential inclusions associated with the interval realization $[T,1-F]$.
5. Investigate intrinsic vague tangent bundles and set-valued vector fields.
6. Extend the theory to stochastic-vague and fuzzy-vague hybrid models.
7. Study stability, Lyapunov functions, bifurcations, and control on compact manifolds.

These directions are proposed research problems rather than claims that the corresponding theories have already been established in this manuscript.

XX. LIMITATIONS AND MATHEMATICAL CAUTIONS

Several limitations are important. First, the pair (T,F) alone does not determine a unique operation for adding, multiplying, or differentiating vague quantities. Such operations must be defined. Second, an interval $[T,1-F]$ is not automatically a tangent vector. Intrinsic interval-valued tangent dynamics requires bundle-valued set-valued analysis. Third, second-order vague equations require a connection. Fourth, numerical projection onto the vague domain changes the numerical method and must be analyzed separately. Finally, applications require a defensible interpretation of truth and falsity degrees; mathematical consistency alone does not establish empirical validity.

XXI. MAIN THEOREMS AND FINDINGS

- A vague scalar field on a smooth manifold can be rigorously represented by differentiable truth and falsity fields T and F with $T+F\leq 1$.
- A vague manifold differential equation can be formulated as an augmented ODE on $M\times D$, where D is the triangular domain of admissible vague values.
- Local existence and uniqueness follow from standard manifold ODE theory under continuity and local Lipschitz hypotheses.

- Boundary conditions on the vague component vector field guarantee positive invariance of the admissible domain.
- Coordinate transformations act on the manifold velocity by the tangent Jacobian, while vague scalar variables remain scalar.
- Classical manifold differential equations are recovered as the zero-width special case T+F=1.
- Gradient-type vague flows can preserve monotonic energy decrease when the vague rate factor is nonnegative.
- A second-order formulation must use a connection or an equivalent geometric acceleration operator.

These results establish a coherent baseline theory but do not by themselves prove superiority over fuzzy, interval, or stochastic approaches. Such comparisons require problem-specific criteria and experiments.

22. Conclusion

This paper introduced a foundational framework for vague differential equations on smooth manifolds. The central idea is to maintain a strict distinction between geometry and information: the state γ(t) is a point of the manifold, its derivative is a tangent vector, and vague information is represented by truth/falsity variables satisfying the admissibility condition T+F≤1. This leads naturally to an augmented dynamical system on M×D.

The framework recovers ordinary manifold differential equations when the vague interval collapses, while permitting residual information to evolve dynamically when the data are incomplete. The circle and sphere examples show that geometric invariants can be preserved exactly while vague coefficients modify the speed or dynamics. Existence, uniqueness, stability, and admissibility results follow from standard differential-equation methods once the model is formulated correctly.

Future work should develop intrinsic set-valued vague tangent bundles, vague differential inclusions, vague connections, vague geodesics, numerical structure preservation, and applications to geometric control and optimization. In particular, a fully intrinsic theory should avoid dependence on arbitrary local frames when interval-valued tangent information itself is treated as a primary object. This provides a substantial mathematical program for developing vague differential geometry and vague dynamical systems.

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