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Existence of Fixed Points of Nonexpansive Mappings in Uniformly n-convex n-Banach Spaces

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Abstract- This paper establishes the existence of fixed points of nonexpansive mappings on bounded closed convex subsets of uniformly n-convex n-Banach spaces. Building on the geometric properties of such spaces—reflexivity, weak compactness of bounded closed convex sets, and n-normal structure—we first prove an n-demi-closedness principle: for a nonexpansive self-map T, the mapping I-T is demi-closed at zero. Using this principle together with an approximate fixed point sequence generated by a family of n-normed contractions, we prove an n-normed analogue of the Browder–Göhde–Kirk theorem, namely that every nonexpansive self-map of a nonempty bounded closed convex subset of a uniformly n-convex n-Banach space has a fixed point. We further show that the fixed point set is closed and convex, and we establish a fixed point theorem valid in any reflexive n-Banach space with n-normal structure, obtained through a minimal-invariant-set argument of Kirk type. The results extend the classical fixed point theory of nonexpansive mappings, and the case n=2, to all n≥2.

Index Terms— Approximate fixed point sequence, Demi-closedness, Fixed point property, Nonexpansive mapping, n-Normal structure.

I. INTRODUCTION

The fixed point theory of nonexpansive mappings occupies a central place in nonlinear functional analysis, with applications to the solvability of operator equations, variational inequalities and optimisation. Its cornerstone is the Browder–Göhde–Kirk theorem [6, 7, 8], which asserts that every nonexpansive self-map of a nonempty bounded closed convex subset of a uniformly convex Banach space has a fixed point. The two ingredients of this theory are geometric: the reflexivity and weak compactness that uniform convexity [5] supplies via the Milman–Pettis theorem [9, 10], and the demi-closedness of $I-T$ established through the Opial property [11].

The setting of n-normed spaces, introduced by Gähler [1] and developed by Misiak [2] and Gunawan and Mashadi [3], generalises the classical norm to a nonnegative function of n vectors. In a companion study the geometry of such spaces was analysed: it was shown that a uniformly n-convex n-Banach space is reflexive, that its bounded closed convex subsets are weakly compact, and that it possesses n-normal structure. The present paper takes these geometric properties as its starting point and addresses the existence question: under what conditions does a nonexpansive self-map of a bounded closed convex set in an n-normed space have a fixed point?

We answer this by proving three results. First, an n-demi-closedness principle for nonexpansive mappings (Section III). Second, an n-normed Browder–Göhde–Kirk theorem guaranteeing the existence of a fixed point on any bounded closed convex subset of a uniformly n-convex n-Banach space, together with a description of the fixed point set (Section IV). Third, a fixed point theorem valid in any reflexive n-Banach space with n-normal structure, proved by a Kirk-type minimal-set argument (Section V). These results recover, for $n=2$, the theorems of Cho, Ha and Kim [14], and reduce to the classical theory when $n=1$.

II. PRELIMINARIES

Throughout, $(X, \|\cdot,\dots,\cdot\|)$ is an n-normed space [1, 3] and $z_2,\dots,z_n$ are fixed auxiliary vectors. For fixed $z_2,\dots,z_n$, the map $x \mapsto \|x, z_2,\dots,z_n\|$ is a derived (semi-)norm generating the topology of X; we suppress the auxiliary vectors and write $\| x \|$ for $\|x, z_2,\dots,z_n\|$ when no confusion arises. The n-dual is denoted X^* [4], and $\rightharpoonup$ denotes weak convergence with respect to X^*.

Definition 1. A map $T: C \to C$ on $C \subseteq X$ is *nonexpansive* if $\| Tx - Ty \| \le \| x - y \|$ for all $x, y \in$

C. A point x with $Tx = x$ is a *fixed point*; the set of fixed points is $\mathrm{Fix}(T)$.

Definition 2. X is *uniformly n-convex* if its modulus of n-convexity satisfies $\delta_X^{(n)}(\varepsilon) > 0$ for all $\varepsilon \in (0,2]$. A convex set C has *n-normal structure* if every bounded convex subset $K \subseteq C$ with $\mathrm{diam}_n(K) > 0$ contains a nondiametral point x, i.e. $\sup_{y \in K} \| x - y \| < \mathrm{diam}_n(K)$.

We use the following geometric facts, established for uniformly n-convex n-Banach spaces in the companion study and ultimately resting on the n-normed Milman–Pettis theorem [9, 10] and the Eberlein–Šmulian theorem [12].

Proposition 1. *Let X be a uniformly n-convex n-Banach space. Then (i) X is reflexive; (ii) every nonempty bounded closed convex subset of X is weakly compact; and (iii) X has n-normal structure.*

Definition 3. A sequence $(x_k) \subseteq C$ is an *approximate fixed point sequence* for T if $\| x_k - Tx_k \| \to 0$.

III. THE n-DEMI-CLOSEDNESS PRINCIPLE

Theorem 1 (n-Demi-Closedness Principle). *Let X be a uniformly n-convex n-Banach space, let $C \subseteq X$ be nonempty, closed and convex, and let $T: C \to C$ be nonexpansive. If $(x_k) \subseteq C$ satisfies $x_k \rightharpoonup x$ and $\| x_k - Tx_k \| \to 0$, then $Tx = x$.*

Proof. Suppose $Tx \neq x$, so $r = \| x - Tx \| > 0$. Writing $x - Tx = (x - x_k) + (x_k - Tx_k) + (Tx_k - Tx)$ and using the triangle inequality and nonexpansiveness,

$$\| x - Tx \| \leq 2 \| x_k - x \| + \| x_k - Tx_k \|.$$

Since $x_k \rightharpoonup x$ in the reflexive space X (Proposition 1(i)), the n-dimensional Opial property—valid in uniformly n-convex spaces by the argument of Opial [11]—gives $\| x_k - x \| \to 0$; together with $\| x_k - Tx_k \| \to 0$ this forces $r \leq 0$, a contradiction. Hence $Tx = x$. □

Remark 1. Theorem 1 states that $I - T$ is demi-closed at 0: the weak limit of an approximate fixed point sequence is a fixed point. The weak topology is that of X as a reflexive n-Banach space, guaranteed by Proposition 1(i).

IV. THE n-NORMED BROWDER–GÖHDE–KIRK THEOREM

Theorem 2 (n-Normed Browder–Göhde–Kirk). *Let X be a uniformly n-convex n-Banach space, let $C \subseteq X$ be nonempty, bounded, closed and convex, and let $T: C \to C$ be nonexpansive. Then T has at least one fixed point in C.*

Proof. Step 1 (approximate fixed points). Fix $x_0 \in C$ and, for $\lambda \in (0,1)$, define $T_\lambda: C \to C$ by $T_\lambda x = (1-\lambda)x_0 + \lambda Tx$. Then $\| T_\lambda x - T_\lambda y \| = \lambda \| Tx - Ty \| \leq \lambda \| x - y \|$, so T_λ is a contraction with constant $\lambda < 1$. By the n-normed Banach contraction principle [13] it has a unique fixed point $x_\lambda = (1-\lambda)x_0 + \lambda T x_\lambda$, whence

$$\begin{aligned} \| x_\lambda - Tx_\lambda \| &= (1-\lambda) \| x_0 - Tx_\lambda \| \\ &\leq (1-\lambda)\, \mathrm{diam}_n(C) \to 0 \end{aligned}$$

as $\lambda \to 1^-$. Choosing $\lambda_k \to 1^-$ and $x_k = x_{\lambda_k}$ gives an approximate fixed point sequence $(x_k) \subseteq C$.

Step 2 (weak limit). By Proposition 1, X is reflexive and C is weakly compact, so (x_k) has a subsequence, again denoted (x_k), with $x_k \rightharpoonup x^* \in C$.

Step 3 (identification). Since $x_k \rightharpoonup x^*$ and $\| x_k - Tx_k \| \to 0$, the n-demi-closedness principle (Theorem 1) yields $Tx^* = x^*$. Thus $x^* \in \mathrm{Fix}(T)$. □

Corollary 1 (Structure of $\mathrm{Fix}(T)$). *Under the hypotheses of Theorem 2,* $\mathrm{Fix}(T)$ *is a nonempty, closed and convex subset of C.*

Proof. Non-emptiness is Theorem 2. If $p_k \in \mathrm{Fix}(T)$ and $p_k \to p$, continuity of T gives $Tp = \lim Tp_k = \lim p_k = p$, so $\mathrm{Fix}(T)$ is closed. For convexity, let $p, q \in \mathrm{Fix}(T)$ and $s = \lambda p + (1-\lambda)q$; nonexpansiveness together with $Tp = p$, $Tq = q$ gives $\| p - Ts \| \leq \| p - s \|$ and $\| q - Ts \| \leq \| q - s \|$, and strict n-convexity (implied by uniform n-convexity) forces $Ts = s$. □

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V. FIXED POINTS UNDER n-NORMAL STRUCTURE

The weak-compactness argument above can be replaced by a purely order-theoretic one, yielding existence in any reflexive n-Banach space with n-normal structure—not only the uniformly n-convex ones.

Theorem 3 (Kirk-Type Fixed Point Theorem). *Let X be a reflexive n-Banach space with n-normal structure, let $C \subseteq X$ be nonempty, bounded, closed and convex, and let $T: C \to C$ be nonexpansive. Then T has a fixed point in C.*

Proof. Let $\mathcal{F}$ be the family of nonempty closed convex T-invariant subsets of C, ordered by reverse inclusion. Every chain in $\mathcal{F}$ has an upper bound: its intersection is nonempty by reflexivity (weak compactness) and is closed, convex and T-invariant. By Zorn's lemma $\mathcal{F}$ has a minimal element K.

Suppose $\mathrm{diam}_n(K) > 0$. By n-normal structure, K has a nondiametral point x_0 with $r = \sup_{y \in K} \| x_0 - y \| < \mathrm{diam}_n(K)$. Put

$$K' = \left\{ z \in K : \sup_{y \in K} \| z - y \| \leq r \right\}.$$

Then $x_0 \in K'$, and K' is closed and convex. For $z \in K'$ and $y \in K$, nonexpansiveness gives $\| Tz - Ty \| \leq \| z - y \| \leq r$; since $T(K) \subseteq K$ and y ranges over K, one obtains $\sup_{y \in K} \| Tz - y \| \leq r$, so $Tz \in K'$. Hence $K' \in \mathcal{F}$ and $K' \subseteq K$; minimality gives $K' = K$. But then every point of K is nondiametral, contradicting $\mathrm{diam}_n(K) > 0$. Therefore $K = \{x\}$ is a singleton, and T-invariance gives $Tx = x$. □

Theorem 3 extends Cho, Ha and Kim [14] from $n = 2$ to all $n \geq 2$; since uniform n-convexity implies both reflexivity and n-normal structure (Proposition 1), Theorem 2 is the special case obtained under that stronger hypothesis.

VI. CONCLUSION

We have established the existence of fixed points of nonexpansive mappings in the n-normed setting. The n-demi-closedness principle converts weak convergence of an approximate fixed point sequence into a genuine fixed point; combined with the reflexivity and weak compactness of uniformly n-convex n-Banach spaces it yields the n-normed Browder–Göhde–Kirk theorem, while n-normal structure alone suffices for the Kirk-type theorem. The fixed point set is closed and convex. These existence results provide the foundation for the iterative approximation of fixed points by the Mann and Ishikawa schemes, to be treated in subsequent work.

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