Five Approaches to Computing *e*: Derivations, Error Bounds and History

LINET MUHATI
Department of Mathematics and Computer Science, University of Eldoret, Eldoret, Kenya

Abstract- ***Euler's number e admits five classical characterisations: as the limit of $(1+1/n)^n$, as the sum of the reciprocals of the factorials, as the base whose exponential function is its own derivative, as the number whose natural logarithm is 1, and through its regular continued fraction $[2;1,2,1,1,4,1,1,6,\ldots]$. This paper gives a self-contained account of the five approaches with complete proofs that they define the same number, and for each approach that yields an algorithm it proves an explicit error bound: $e-(1+1/n)^n = e/(2n)+O(n^{-2})$, a remainder below $1/(n\cdot n!)$ for the n-th partial sum of the series, and $|e-p_k/q_k|<1/q_k^2$ for the continued fraction convergents. It shows that the limit is exactly Euler's method applied to $y'=y$, that the trapezoidal rule yields the second-order approximation $\left((2n+1)/(2n-1)\right)^n$, and that Newton's method turns the integral definition into a quadratically convergent scheme. The methods are compared by the work needed per correct digit, and the historical account is corrected against primary and secondary sources.***

Index Terms- ***Continued fraction, Convergence rate, Euler's number, Exponential function, Numerical approximation***

I. INTRODUCTION

The number $e = 2.718281828\ldots$ is the base of the natural logarithm and of the exponential function, the unique function equal to its own derivative and taking the value 1 at 0. It first appeared implicitly in two seventeenth-century problems: the area under the hyperbola $y=1/t$, whose logarithmic character was recognised by Grégoire de Saint-Vincent and Alphonse Antonio de Sarasa in the 1640s [1], and the problem of continuously compounded interest posed by Jacob Bernoulli in 1690 [2]. Mercator's *Logarithmotechnia* of 1668 gave the series for $\ln(1+x)$ and introduced the name "natural logarithm" [3]. The constant itself was identified and named by Euler, who in the *Introductio in analysin infinitorum* of 1748 established its series and its connection with logarithms [4], and who in a memoir presented in 1737 obtained its continued fraction and deduced that e is irrational [5]. Hermite proved in 1873 that e is transcendental [6]. Historical accounts are given by Coolidge [7] and Maor [8].

This paper treats five characterisations of e: the limit of $(1+1/n)^n$, the factorial series, the self-derivative property of e^x, the integral definition of the logarithm, and the regular continued fraction. The material is classical, and the aim of the paper is expository. Its contribution is to present the five approaches in one place with complete arguments and quantitative conclusions. Specifically, we

1. prove that the five characterisations define the same number, giving the limit interchanges and existence arguments that are often left informal;
2. prove an explicit error estimate for each approach that yields an algorithm, and verify it against computed values;
3. show that the compound-interest limit is exactly Euler's method for $y'=y$, that the trapezoidal rule gives a second-order approximation, and that Newton's method converts the integral definition into a quadratically convergent algorithm;
4. compare the algorithms by the work required per correct decimal digit rather than by the number of terms; and
5. correct several historical attributions that recur in expository accounts.

Section II treats the limit, Section III the series, Section IV the self-derivative property and Section V the integral definition. Section VI studies the continued fraction, Section VII compares the algorithms and Section IX concludes. All numerical values were computed in 30-digit arithmetic and are rounded in the last digit shown.

II. APPROACH 1: THE COMPOUND-INTEREST LIMIT

A. Existence of the limit

Bernoulli asked how much a unit of capital grows in one year at an annual rate of 100% when interest is compounded n times, at a rate of $1/n$ per period [2]. The amount is

$$a_n = \left(1+\frac{1}{n}\right)^n,$$

and Bernoulli observed that it increases with n but remains bounded, between 2 and 3 [7, 8]. We give a complete proof that a_n converges, together with an explicit error bound.

Theorem 2.1. Let $a_n = (1+1/n)^n$ and $c_n = (1+1/n)^{n+1}$ for $n \geq 1$. Then (a_n) is strictly increasing, (c_n) is strictly decreasing, and both converge to a common limit e. Moreover

© SEP 2026 | IRE Journals | Volume 10 Issue 3 | ISSN: 2456-8880
DOI: https://doi.org/10.64388/IREV10I3-1723438

$$a_n < e < c_n \qquad \text{and} \qquad 0 < e - a_n < \frac{e}{n}. \qquad (1)$$

Proof. Apply the inequality of arithmetic and geometric means to the $n+1$ positive numbers $1+\frac{1}{n}$ (taken n times) and 1. They are not all equal, so

$$\left(1+\frac{1}{n}\right)^{\frac{n}{n+1}} < \frac{n\left(1+\frac{1}{n}\right)+1}{n+1} = 1+\frac{1}{n+1},$$

and raising to the power $n+1$ gives $a_n < a_{n+1}$. Similarly, applying the inequality to $1-\frac{1}{n+1}$ (taken $n+1$ times) and 1 gives

$$\left(1-\frac{1}{n+1}\right)^{n+1} < \left(1-\frac{1}{n+2}\right)^{n+2}.$$

The left side is $1/c_n$ and the right side is $1/c_{n+1}$, so $c_{n+1} < c_n$. Since $a_n < c_n \le c_1 = 4$, the sequence (a_n) is increasing and bounded, and (c_n) is decreasing and bounded below by $a_1 = 2$; both therefore converge. Their difference is $c_n - a_n = a_n/n \to 0$, so the limits coincide. Call the common limit e. Then $a_n < e < c_n$ by strict monotonicity, and $e - a_n < c_n - a_n = a_n/n < e/n$. ▫

B. The rate of convergence

The bound $e - a_n < e/n$ is of the correct order, and the leading constant can be found exactly.

Theorem 2.2. As $n \to \infty$,

$$e - \left(1+\frac{1}{n}\right)^n = \frac{e}{2n} - \frac{11e}{24n^2} + O(n^{-3}).$$

Proof. For $n \ge 1$ the logarithmic series gives

$$n\ln\left(1+\frac{1}{n}\right) = 1 - \frac{1}{2n} + \frac{1}{3n^2} + O(n^{-3}).$$

Exponentiating, with $\exp(x) = 1 + x + \frac{x^2}{2} + O(x^3)$ and $x = -\frac{1}{2n} + \frac{1}{3n^2} + O(n^{-3})$,

$$\begin{aligned} a_n &= e\left(1 - \frac{1}{2n} + \frac{1}{3n^2} + \frac{1}{8n^2} + O(n^{-3})\right) \\ &= e - \frac{e}{2n} + \frac{11e}{24n^2} + O(n^{-3}). \end{aligned}$$

▫

Table 1 confirms Theorem 2.2: the ratio of the actual error to $e/(2n)$ tends to 1. The error is roughly $1.36/n$, so each additional correct decimal digit costs a tenfold increase in n, and d correct digits require n of the order of $e \cdot 10^d$. This quantifies the slow convergence of the limit.

The limit $a_n = (1+1/n)^n$ and its error.

n	a_n	$e - a_n$	$\frac{e-a_n}{e/(2n)}$
1	2.000000	7.18×10^{-1}	0.528
10	2.593742	1.25×10^{-1}	0.916
10^2	2.704814	1.35×10^{-2}	0.991
10^3	2.716924	1.36×10^{-3}	0.9991
10^4	2.718146	1.36×10^{-4}	0.99991
10^5	2.718268	1.36×10^{-5}	0.999991
10^6	2.718280	1.36×10^{-6}	0.999999

C. The limit as Euler's method

The compound-interest limit has a natural interpretation in numerical analysis. Consider the initial value problem

$$y' = y, \qquad y(0) = 1, \qquad (2)$$

whose solution is $y(t) = e^t$, and approximate $y(1) = e$ with n steps of length $h = 1/n$.

Proposition 2.3. Explicit Euler's method $y_{k+1} = y_k + hy_k$ applied to (2) with $h = 1/n$ gives $y_n = (1+1/n)^n = a_n$. The trapezoidal rule $y_{k+1} = y_k + \frac{h}{2}(y_k + y_{k+1})$ gives

$$y_n = b_n := \left(\frac{2n+1}{2n-1}\right)^n.$$

Proof. Euler's method gives $y_{k+1} = (1+h)y_k$, hence $y_n = (1+1/n)^n$. The trapezoidal rule gives $\left(1-\frac{h}{2}\right)y_{k+1} = \left(1+\frac{h}{2}\right)y_k$, hence $y_{k+1} = \frac{2+h}{2-h}y_k = \frac{2n+1}{2n-1}y_k$ when $h = 1/n$. ▫

Proposition 2.3 explains Theorem 2.2: Euler's method has global error of order h, while the trapezoidal rule is of order h^2 [9]. The second-order approximation b_n also appears among the closed-form approximations to e collected by Brothers and Knox [10]. Its error can be found exactly as in Theorem 2.2.

Theorem 2.4. For every $n \ge 1$ one has $b_n > e$, and

$$b_n - e = \frac{e}{12n^2} + O(n^{-4}) \qquad (n \to \infty).$$

Proof. With $u = \frac{1}{2n}$ we have $\frac{2n+1}{2n-1} = \frac{1+u}{1-u}$, and

$$\begin{aligned} n\ln\frac{1+u}{1-u} &= 2n\left(u + \frac{u^3}{3} + \frac{u^5}{5} + \cdots\right) \\ &= 1 + \frac{1}{12n^2} + O(n^{-4}). \end{aligned}$$

Every term of the series is positive, so the left side exceeds $2nu = 1$ and $b_n > e$. Exponentiating gives $b_n = e\left(1 + \frac{1}{12n^2} + O(n^{-4})\right)$. ▫

The trapezoidal approximation $b_n = \left((2n+1)/(2n-1)\right)^n$.

n	b_n	$b_n - e$	$e/(12n^2)$
1	3.000000000	2.82×10^{-1}	2.27×10^{-1}
10	2.720551414	2.27×10^{-3}	2.27×10^{-3}
10^2	2.718304481	2.27×10^{-5}	2.27×10^{-5}
10^3	2.718282055	2.27×10^{-7}	2.27×10^{-7}

Table 2 shows that b_{1000} is correct to six decimal places, an accuracy for which a_n needs n of the order of 10^6. Thus a change of discretisation, rather than of

© SEP 2026 | IRE Journals | Volume 10 Issue 3 | ISSN: 2456-8880
DOI: https://doi.org/10.64388/IREV10I3-1723438

definition, improves the limit method from first to second order.

III. APPROACH 2: THE FACTORIAL SERIES

A. Equality with the limit

Let $s_n = \sum_{k=0}^{n} 1/k!$. Since $1/k! \leq 2^{1-k}$ for $k \geq 1$, we have $s_n < 1 + \sum_{k\geq 1} 2^{1-k} = 3$, so the increasing sequence (s_n) converges. The usual derivation expands a_n by the binomial theorem and lets $n \to \infty$ term by term; because the number of terms grows with n, this interchange must be justified. The following squeeze argument, which follows Rudin [11], does so.

Theorem 3.1. $e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = \sum_{k=0}^{\infty} \frac{1}{k!}$.

Proof. By the binomial theorem,

$$a_n = \sum_{k=0}^{n} \frac{1}{k!} \prod_{j=0}^{k-1} \left(1 - \frac{j}{n}\right). \qquad (3)$$

Each product lies in $[0,1]$, so $a_n \leq s_n$ and hence $e \leq \lim s_n$. Conversely, fix m and let $n > m$. Dropping the terms with $k > m$ from (3) gives

$$a_n \geq \sum_{k=0}^{m} \frac{1}{k!} \prod_{j=0}^{k-1} \left(1 - \frac{j}{n}\right).$$

The right side has a fixed number of terms, each tending to $1/k!$ as $n \to \infty$, so $e \geq s_m$. Letting $m \to \infty$ gives $e \geq \lim s_m$. ▫

B. The remainder

Theorem 3.2. For every $n \geq 1$,

$$\frac{1}{(n+1)!} < e - s_n < \frac{1}{n \cdot n!}.$$

Proof. The lower bound is the first omitted term. For the upper bound, since $(n+2)(n+3)\cdots(n+1+j) > (n+1)^j$ for $j \geq 1$,

$$\begin{aligned} e - s_n &= \frac{1}{(n+1)!}\left(1 + \frac{1}{n+2} + \frac{1}{(n+2)(n+3)} + \cdots\right) \\ &< \frac{1}{(n+1)!} \sum_{j\geq 0} \frac{1}{(n+1)^j} \\ &= \frac{1}{(n+1)!} \cdot \frac{n+1}{n} = \frac{1}{n \cdot n!}. \end{aligned}$$

▫

Table 3 shows that the upper bound is sharp to within a few per cent. It also settles how many terms are needed for a given accuracy. The partial sum s_8, which has nine terms, equals 2.7182788 and is correct to only five decimal places; six correct decimals, and an error below 10^{-6}, require s_9, which has ten terms. In general the number of correct digits grows like $\log_{10} n! \approx n \log_{10}(n/e)$, which is faster than any power of n.

Partial sums $s_n = \sum_{k=0}^{n} 1/k!$ and the bound of Theorem 3.2.

n	s_n	$e - s_n$	$1/(n \cdot n!)$
4	2.708333333	9.95×10^{-3}	1.04×10^{-2}
6	2.718055556	2.26×10^{-4}	2.31×10^{-4}
8	2.718278770	3.06×10^{-6}	3.10×10^{-6}
9	2.718281526	3.03×10^{-7}	3.06×10^{-7}
10	2.718281801	2.73×10^{-8}	2.76×10^{-8}
12	2.718281828	1.73×10^{-10}	1.74×10^{-10}

C. Irrationality

The remainder estimate gives Fourier's proof that e is irrational [12].

Theorem 3.3. The number e is irrational.

Proof. Suppose $e = p/q$ with p, q positive integers. Then $N = q!\,(e - s_q)$ is an integer, because $q!\,e = p\,(q-1)!$ and $q!\,s_q$ are integers. By Theorem 3.2, $0 < N < \frac{q!}{q \cdot q!} = \frac{1}{q} \leq 1$, which is impossible. ▫

D. High-precision evaluation

For fixed-precision arithmetic a stored constant suffices, but when many digits are wanted the series is the method of choice. Summing s_n term by term costs one division per term at full precision. Binary splitting instead groups the terms recursively into a single fraction P/Q with integer numerator and denominator and performs one final division, which reduces the cost to nearly linear in the number of digits, up to logarithmic factors; Haible and Papanikolaou give the method and its analysis for general series of rational numbers [13].

IV. APPROACH 3: THE SELF-DERIVATIVE PROPERTY

A. Avoiding circularity

The statement that e is the unique base a for which $\frac{d}{dx}a^x = a^x$ is often justified by writing $\frac{d}{dx}a^x = a^x L(a)$ with $L(a) = \lim_{h\to 0}(a^h - 1)/h$ and appealing to continuity of L. This presupposes that a^x is defined for real x, that the limit exists, and that L is continuous and increasing; since in fact $L(a) = \ln a$, the argument is circular unless the exponential or the logarithm has been constructed first. We therefore construct the exponential from the series, following [11, 14].

For $x \in \mathbb{R}$ define

$$E(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

The series converges absolutely for all x by the ratio test. Multiplying the series for $E(x)$ and $E(y)$ and collecting terms by the binomial theorem gives $E(x+y) = E(x)E(y)$ [11]. A power series may be differentiated term by term inside its interval of convergence, so $E'(x) = E(x)$. By Theorem 3.1, $E(1) = e$. Since $E(x)E(-x) = E(0) = 1$ and $E(x) > 0$ for $x \geq 0$, E is positive everywhere, hence strictly increasing, and $E(p/q) = e^{p/q}$ for rational p/q. We therefore define $e^x = E(x)$ for all real x.

B. Uniqueness

Theorem 4.1. If $f: \mathbb{R} \to \mathbb{R}$ is differentiable with $f' = f$ and $f(0) = 1$, then $f(x) = e^x$ for all x.
Proof. Let $g = f/E$, which is defined because E has no zeros. Then $g' = (f'E - fE')/E^2 = 0$, so g is constant, and $g(0) = 1$. ▫
Since E is a continuous strictly increasing bijection of $\mathbb{R}$ onto $(0, \infty)$, it has an inverse, the natural logarithm ln. For $a > 0$ and real x define $a^x = E(x\ln a)$, which agrees with the elementary definition for rational x. The chain rule gives

$$\frac{d}{dx} a^x = (\ln a)\, a^x,$$

so the slope of a^x at $x = 0$ is $L(a) = \ln a$, a continuous strictly increasing function of a equal to 1 exactly when $a = E(1) = e$. This is the rigorous form of the self-derivative characterisation. The values $\ln 2 = 0.693147$ and $\ln 3 = 1.098612$ show again that $2 < e < 3$.
The self-derivative property is a definition, not an algorithm. It becomes one when the equation $y' = y$ is solved numerically, which returns us to Section II: Euler's method gives $(1 + 1/n)^n$ and the trapezoidal rule gives $\left((2n+1)/(2n-1)\right)^n$.

V. APPROACH 4: THE INTEGRAL DEFINITION

A. The logarithm as an area

For $x > 0$ define

$$\ell(x) = \int_1^x \frac{dt}{t}.$$

Then ℓ is continuous and differentiable with $\ell'(x) = 1/x > 0$, so it is strictly increasing, and $\ell(1) = 0$. The substitution $t = xs$ gives $\ell(xy) = \ell(x) + \ell(y)$, so $\ell(2^m) = m\,\ell(2) \to \infty$ and $\ell(2^{-m}) \to -\infty$. By the intermediate value theorem there is exactly one number $\varepsilon > 0$ with $\ell(\varepsilon) = 1$ [15].
Theorem 5.1. The number ε defined by $\int_1^\varepsilon d\,t/t = 1$ equals e, and $\ell = \ln$.
Proof. The function ℓ is a strictly increasing bijection of $(0, \infty)$ onto $\mathbb{R}$. Its inverse F satisfies $F(0) = 1$ and, by the inverse function rule, $F'(y) = 1/\ell'\left(F(y)\right) = F(y)$. By Theorem 4.1, $F = E$, so $\ell = \ln$ and $\varepsilon = F(1) = E(1) = e$. ▫
Geometrically, e is the point beyond 1 at which the area under the hyperbola $y = 1/t$ reaches 1. The logarithmic behaviour of hyperbolic areas was recognised by Grégoire de Saint-Vincent and interpreted as a logarithm by de Sarasa in the 1640s [1], and Mercator obtained the series $\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$ in 1668 [3]. None of these authors singled out the value e; the base of the hyperbolic logarithm was identified in the eighteenth century and named by Euler [4, 7].

B. Newton's method

The integral definition becomes an algorithm when the equation $\ln x = 1$ is solved numerically. Newton's method for $g(x) = \ln x - 1$ gives the iteration

$$x_{k+1} = x_k - \frac{\ln x_k - 1}{1/x_k} = x_k\,(2 - \ln x_k). \qquad (4)$$

Proposition 5.2. Let $0 < x_0 < e^2$. Then $x_1 \leq e$, the iterates $x_1, x_2, \ldots$ increase to e, and the relative errors $\delta_k = x_k/e - 1$ satisfy $\delta_{k+1} = -1/2\,\delta_k^2 + O(\delta_k^3)$.
Proof. The function $\phi(x) = x(2 - \ln x)$ has $\phi'(x) = 1 - \ln x$, so it increases on $(0, e]$, decreases on $[e, \infty)$ and attains its maximum $\phi(e) = e$. Hence $x_1 = \phi(x_0) \leq e$, and $x_1 > 0$ because $x_0 < e^2$. For $0 < x < e$ we have $\phi(x) > x$ because $2 - \ln x > 1$, so the iterates increase and are bounded by e; their limit is a fixed point of ϕ in $(0, e]$, which can only be e. Finally, writing $x_k = e(1 + \delta_k)$,

$$\begin{aligned} x_{k+1} &= e(1+\delta_k)\left(1 - \ln(1+\delta_k)\right) \\ &= e(1+\delta_k)\left(1 - \delta_k + 1/2\,\delta_k^2 + O(\delta_k^3)\right) \\ &= e\left(1 - 1/2\,\delta_k^2 + O(\delta_k^3)\right). \end{aligned}$$

▫

Newton's method (4) for $\ln x = 1$ with $x_0 = 2$.

k	x_k	$e - x_k$
0	2.000000000000	7.18×10^{-1}
1	2.613705638880	1.05×10^{-1}
2	2.716243926356	2.04×10^{-3}
3	2.718281064358	7.64×10^{-7}
4	2.718281828459	1.07×10^{-13}
5	2.718281828459	2.12×10^{-27}

Table 4 shows the number of correct digits roughly doubling at each step. Each step, however, requires $\ln x_k$ to the target precision, for instance by quadrature or by the series of $\ln\frac{1+u}{1-u}$, so the work lies in evaluating the logarithm. The method is therefore of theoretical rather than practical interest for computing e itself, but it shows that each of the analytic definitions yields a convergent algorithm.

VI. APPROACH 5: THE CONTINUED FRACTION

A. Continued fractions and convergents

A regular continued fraction $[a_0; a_1, a_2, \ldots]$ with integer a_0 and positive integers $a_1, a_2, \ldots$ has convergents $p_k/q_k = [a_0; a_1, \dots, a_k]$, given by the recurrences

$$p_k = a_k p_{k-1} + p_{k-2}, \qquad q_k = a_k q_{k-1} + q_{k-2}, \qquad (5)$$

with $p_{-1} = 1$, $p_{-2} = 0$, $q_{-1} = 0$, $q_{-2} = 1$. The following facts are standard [16]: $p_k q_{k-1} - p_{k-1} q_k = (-1)^{k-1}$, so the fractions p_k/q_k are in lowest terms; if $x = [a_0; a_1, \dots]$ is infinite, the convergents lie alternately below and above x and

$$\left| x - \frac{p_k}{q_k} \right| < \frac{1}{q_k q_{k+1}} < \frac{1}{q_k^2}; \qquad (6)$$

and each convergent is a best approximation, in the sense that no fraction with denominator at most q_k is closer to x. Every irrational number has a unique infinite regular continued fraction, and by a theorem of Euler and Lagrange the expansion is eventually periodic if and only if x is a quadratic irrational [16].

B. The expansion of e

Theorem 6.1 (Euler). The regular continued fraction of e is

$$e = [2; 1,2,1,1,4,1,1,6,1,1,8, \dots],$$

that is, $a_0 = 2$ and, for $m \geq 1$, $a_{3m-1} = 2m$ and $a_{3m-2} = a_{3m} = 1$. Written out,

$$e = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{1 + \cdots}}}}}}.$$

Euler obtained the expansion in a memoir presented in 1737, by relating continued fractions to a Riccati differential equation [5]. A short modern proof, based on integrals of the type used by Hermite, is due to Cohn [17]; Osler gives a direct proof of the more general expansion of $e^{1/M}$ [18]. The expansion is not derived from the factorial series, although both can be obtained from the same analytic information about e^x. Table 5 lists the first convergents as exact fractions together with their errors. The signs alternate, as they must, and every error is smaller than $1/q_k^2$, in accordance with (6). The convergent $2721/1001$ is the first that is correct to six decimal places.

Convergents p_k/q_k of the continued fraction of e.

k	a_k	p_k/q_k	$e - p_k/q_k$	$1/q_k^2$
0	2	$2/1$	$+7.18 \times 10^{-1}$	1
1	1	$3/1$	-2.82×10^{-1}	1
2	2	$8/3$	$+5.16 \times 10^{-2}$	1.11×10^{-1}
3	1	$11/4$	-3.17×10^{-2}	6.25×10^{-2}
4	1	$19/7$	$+4.00 \times 10^{-3}$	2.04×10^{-2}
5	4	$87/32$	-4.68×10^{-4}	9.77×10^{-4}
6	1	$106/39$	$+3.33 \times 10^{-4}$	6.57×10^{-4}
7	1	$193/71$	-2.80×10^{-5}	1.98×10^{-4}
8	6	$1264/465$	$+2.26 \times 10^{-6}$	4.62×10^{-6}
9	1	$1457/536$	-1.75×10^{-6}	3.48×10^{-6}
10	1	$2721/1001$	$+1.10 \times 10^{-7}$	9.98×10^{-7}
11	8	$23225/8544$	-6.75×10^{-9}	1.37×10^{-8}

C. Arithmetic consequences

Theorem 6.1 gives a second proof of Theorem 3.3, since the expansion is infinite; this was Euler's own argument [5]. Because the partial quotients $2m$ are unbounded, the expansion is not eventually periodic, so by the Euler–Lagrange theorem e is not a quadratic irrational. The continued fraction does not, however, establish transcendence: that e satisfies no polynomial equation with integer coefficients was proved by Hermite by a different method [6]. The regularity of the expansion of e contrasts with the case of π, for whose regular continued fraction no pattern is known.

VII. COMPARISON OF THE ALGORITHMS

Comparing methods by the number of terms, as is often done, is misleading, because a term costs very different amounts of work in different methods. We compare them instead by the parameter needed for d correct decimal places, meaning an error below $\frac{1}{2} \cdot 10^{-d}$, and by the arithmetic that parameter entails. Table 6 gives the minimal parameter for $d = 6$, 12 and 24, computed in 80-digit arithmetic.

Minimal parameter for d correct decimal places (n = index, k = convergent, Newton started at $x_0 = 2$).

Method	$d = 6$	$d = 12$	$d = 24$
Limit a_n (n)	2.7×10^6	2.7×10^{12}	2.7×10^{24}
Trapezoid b_n (n)	674	6.7×10^5	6.7×10^{11}
Series s_n (n)	9	15	24
Cont. fraction (k)	10	16	28
Newton (steps)	4	4	5

The table reflects the error estimates proved above.

1. *Limit.* By Theorem 2.2 the error is about $e/(2n)$, so d digits need $n \approx e \cdot 10^d$; the computed values are the first digits of e times 10^d. Evaluating

$(1+1/n)^n$ by repeated squaring takes about $\log_2 n \approx 3.3d$ multiplications, but since $1/n$ is of size 10^{-d} the arithmetic must be carried with roughly $2d$ digits to avoid losing the result to rounding.

2. *Trapezoid.* By Theorem 2.4 the error is about $e/(12n^2)$, so $n \approx (e \cdot 10^d/6)^{1/2}$ and the number of squarings is halved.
3. *Series.* By Theorem 3.2, n terms give roughly $\log_{10}(n \cdot n!)$ digits. Each term is obtained from the previous one by a division by a small integer, so the cost is about n short divisions and additions; with binary splitting the cost becomes nearly linear in d [13].
4. *Continued fraction.* By (6) the k-th convergent gives about $2\log_{10} q_k$ digits. The convergents are computed exactly in integer arithmetic by (5), at the cost of two integer multiplications and additions per step and one final division. The growth of q_k is comparable to that of the factorials, so the number of steps is of the same order as the number of series terms.
5. *Newton.* By Proposition 5.2 the number of correct digits doubles at each step, so few steps are needed, but each step requires a logarithm evaluated to the full precision, which is itself a series computation.

The series and the continued fraction therefore gain correct digits faster than any power of the parameter, the limit and its trapezoidal variant only logarithmically. For practical high-precision computation the series, evaluated by binary splitting, is the method of choice [13]; the continued fraction is the method of choice when the goal is the best rational approximation with a given denominator.

VIII. CONNECTIONS BETWEEN THE APPROACHES

The five characterisations are linked by the results proved above, and the links are summarised here.

1. *Limit and series.* Theorem 3.1 shows that $(1+1/n)^n$ and $\sum 1/k!$ have the same limit, and the binomial expansion (3) shows how the partial sums dominate the terms of the limit.
2. *Series and self-derivative.* The series defines the function $E(x) = \sum x^k/k!$, which satisfies $E' = E$, $E(0) = 1$ and $E(1) = e$, and Theorem 4.1 shows that it is the only such function.
3. *Self-derivative and integral.* The inverse of the area function $\ell(x) = \int_1^x dt/t$ satisfies the same differential equation, so by Theorem 5.1 the integral definition gives the same number.
4. *Self-derivative and limit.* Solving $y' = y$ by Euler's method reproduces the compound-interest limit exactly, and the trapezoidal rule gives the second-order approximation of Theorem 2.4 (Proposition 2.3).
5. *Integral and Newton's method.* Solving $\ln x = 1$ by Newton's method turns the integral definition into the quadratically convergent iteration of Proposition 5.2.
6. *Continued fraction.* The continued fraction rests on an independent analytic argument (Theorem 6.1). Its agreement with the series is confirmed numerically in Tables 3 and 5, and both lead to proofs that e is irrational (Theorem 3.3 and Section VI).

IX. CONCLUSION

We have given a unified account of five classical characterisations of e with complete proofs that they agree. For the approaches that yield algorithms we proved explicit error estimates: $e - (1+1/n)^n = e/(2n) + O(n^{-2})$ with the rigorous bound $e - (1+1/n)^n < e/n$; $b_n - e = e/(12n^2) + O(n^{-4})$ for the trapezoidal approximation $b_n = \left((2n+1)/(2n-1)\right)^n$; $1/(n+1)! < e - s_n < 1/(n \cdot n!)$ for the partial sums of the series; $|e - p_k/q_k| < 1/q_k^2$ for the continued fraction convergents; and quadratic convergence of Newton's method for $\ln x = 1$. The limit was identified with Euler's method for $y' = y$, which explains its first-order convergence and suggests the second-order improvement. A comparison by the parameter required for a given number of correct digits shows that the series and the continued fraction gain digits faster than any power of their parameter, whereas the limit gains them only logarithmically; for high-precision work the series, evaluated by binary splitting, is the method of choice, and the continued fraction gives the best rational approximations. The numerical tables correct several values that recur in expository accounts, notably that ten terms of the series, not nine, are needed for six correct decimal places, and the historical account attributes the hyperbolic logarithm, the compound-interest problem, the naming of e, its continued fraction and its irrationality and transcendence to their correct sources.

REFERENCES

[1] R. P. Burn, "Alphonse Antonio de Sarasa and logarithms," *Historia Math.*, vol. 28, no. 1, pp. 1–17, 2001. ScienceDirect

[2] J. Bernoulli, "Quaestiones nonnullae de usuris, cum solutione problematis de sorte alearum," *Acta Eruditorum*, pp. 219–223, 1690.

[3] N. Mercator, *Logarithmotechnia*. London, U.K., 1668.

[4] L. Euler, *Introduction to Analysis of the Infinite*, Book I, J. D. Blanton, Trans. New York, NY, USA: Springer, 1988. Springer

[5] L. Euler, "An essay on continued fractions," M. F. Wyman and B. F. Wyman, Trans., *Math. Systems Theory*, vol. 18, no. 1, pp. 295–328, 1985. Springer

[6] C. Hermite, "Sur la fonction exponentielle," *C. R. Acad. Sci. Paris*, vol. 77, pp. 18–24, 74–79, 226–233, 285–293, 1873.

[7] J. L. Coolidge, "The number e," *Amer. Math. Monthly*, vol. 57, no. 9, pp. 591–602, 1950. Taylor & Francis

[8] E. Maor, *e: The Story of a Number*. Princeton, NJ, USA: Princeton Univ. Press, 1994.

[9] E. Hairer, S. P. Nørsett, and G. Wanner, *Solving Ordinary Differential Equations I: Nonstiff Problems*, 2nd ed., ser. Springer Series in Computational Mathematics, vol. 8. Berlin, Germany: Springer, 1993. Springer

[10] H. J. Brothers and J. A. Knox, "New closed-form approximations to the logarithmic constant e," *Math. Intelligencer*, vol. 20, no. 4, pp. 25–29, 1998. Springer

[11] W. Rudin, *Principles of Mathematical Analysis*, 3rd ed. New York, NY, USA: McGraw-Hill, 1976.

[12] M. Aigner and G. M. Ziegler, *Proofs from THE BOOK*, 6th ed. Berlin, Germany: Springer, 2018. Springer

[13] B. Haible and T. Papanikolaou, "Fast multiprecision evaluation of series of rational numbers," in *Algorithmic Number Theory (ANTS-III)*, ser. Lecture Notes in Computer Science, vol. 1423. Berlin, Germany: Springer, 1998, pp. 338–350. Springer

[14] M. Spivak, *Calculus*, 4th ed. Houston, TX, USA: Publish or Perish, 2008.

[15] T. M. Apostol, *Calculus*, Vol. I, 2nd ed. New York, NY, USA: Wiley, 1967.

[16] C. D. Olds, *Continued Fractions*, ser. New Mathematical Library, vol. 9. Washington, DC, USA: Mathematical Association of America, 1963. Cambridge

[17] H. Cohn, "A short proof of the simple continued fraction expansion of e," *Amer. Math. Monthly*, vol. 113, no. 1, pp. 57–62, 2006. Taylor & Francis

[18] T. J. Osler, "A proof of the continued fraction expansion of e^(1/M)," *Amer. Math. Monthly*, vol. 113, no. 1, pp. 62–66, 2006. Taylor & Francis