

Maximal Ideals and the Numerical Range in Banach Algebras: A Survey

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Abstract- *We survey the relationship between the maximal ideal space of a commutative Banach algebra and the numerical range of its elements and of operators on Banach spaces. The Gelfand transform identifies the spectrum of an element with the range of its transform on the character space, and since every character is a state, the spectrum lies inside the algebra numerical range. Starting from this inclusion we describe how the Šilov and Choquet boundaries, Hermitian elements and the Vidav–Palmer theorem, and the ideal and M-ideal structure of an algebra interact with numerical ranges. We then review results obtained since 1990 on the Bochner–Schoenberg–Eberlein property, numerical radius inequalities, closedness of the numerical range of compact operators, non-commutative boundaries of operator systems, and the Crouzeix–Palencia theorem that the numerical range is a $(1 + \sqrt{2})$ -spectral set. Every result is stated with its hypotheses and with a source, several standard claims are illustrated by explicit examples and counterexamples, and the paper closes with open problems that arise from this literature.*

Index Terms- *Banach algebra, Gelfand transform, Maximal ideal space, Numerical range, Šilov boundary*

I. INTRODUCTION

The theory of commutative Banach algebras was created by Gelfand [1], whose central idea was to attach to a commutative unital Banach algebra A the set of its maximal ideals, to identify this set with the space $\Delta(A)$ of nonzero multiplicative linear functionals, and to represent each element $a \in A$ as a continuous function $\hat{a}$ on the compact Hausdorff space $\Delta(A)$. Under this representation the spectrum of a becomes the range of $\hat{a}$, so that questions about

invertibility and spectra are converted into questions about functions on a topological space. Modern accounts of the theory are given by Kaniuth [2] and Palmer [3].

The numerical range has a different origin. Toeplitz [4] introduced the set $W(T) = \{Tx, x : \|x\| = 1\}$ for a matrix T and showed that its boundary is convex; Hausdorff [5] completed the proof that $W(T)$ itself is convex. Lumer [6] extended the definition to operators on Banach spaces by means of semi-inner products, and Bonsall and Duncan [7, 8] developed the algebra numerical range $V(A, a)$, defined through the states of a unital Banach algebra, into a general theory. A recent monograph on the Hilbert space case is that of Wu and Gau [9].

The two notions meet at a single observation: every character of a unital Banach algebra is a state. Consequently the spectrum, which is the range of the Gelfand transform, is contained in the numerical range, and the numerical range can be regarded as a convex enlargement of the spectrum that also records metric information about the algebra. This paper surveys what is known about this interplay. We aim to give a single account in which the commutative theory of maximal ideals and boundaries, the numerical range theory of Bonsall and Duncan, and a selection of more recent results are presented with precise hypotheses, and in which the limits of the standard statements are marked by explicit examples.

The paper is organised as follows. Section II fixes notation and collects the basic facts on characters, the Gelfand transform and numerical ranges, including the sharp comparison between the norm and the numerical radius. Section III treats the Gelfand representation and the Šilov and Choquet boundaries. Section IV studies the numerical range in relation to the character

space, Hermitian elements and the Vidav–Palmer theorem. Section V reviews developments since 1990: the Bochner–Schoenberg–Eberlein property, numerical radius inequalities, the numerical range of compact and of unbounded operators, M-ideals, and non-commutative boundaries. Section VI discusses applications to functional calculus, including the Crouzeix conjecture, and to Gleason parts. Section VII states open problems and concludes.

Throughout, algebras and spaces are over the complex field $\mathbb{C}$. When a statement requires a unit, commutativity or semisimplicity, the hypothesis is stated explicitly.

II. PRELIMINARIES

A. Characters and the Gelfand transform

A *Banach algebra* is a complex Banach space A with an associative bilinear multiplication satisfying $\|ab\| \leq \|a\| \|b\|$. It is *unital* if it has an identity e , in which case we assume $\|e\| = 1$. A *character* of A is a nonzero multiplicative linear functional $\varphi: A \rightarrow \mathbb{C}$, and the set of characters is denoted $\Delta(A)$.

Let A be commutative and unital. Every character satisfies $\varphi(e) = 1$ and $|\varphi(a)| \leq r(a) \leq \|a\|$, where $r(a)$ is the spectral radius, so every character is continuous with $\|\varphi\| = 1$. By the Gelfand–Mazur theorem the map $\varphi \mapsto \ker \varphi$ is a bijection from $\Delta(A)$ onto the set of maximal ideals of A ; in particular every maximal ideal is closed and has codimension one [2]. For this reason $\Delta(A)$ is called the *maximal ideal space*, or *character space*, of A .

Endowed with the relative weak* topology of the dual space A^* , the set $\Delta(A)$ is a compact Hausdorff space. The *Gelfand transform* of $a \in A$ is the function $\hat{a}: \Delta(A) \rightarrow \mathbb{C}$, $\hat{a}(\varphi) = \varphi(a)$, and the *Gelfand map* $\Gamma: a \mapsto \hat{a}$ is a homomorphism from A into $C(\Delta(A))$ with

$$\sigma(a) = \hat{a}(\Delta(A)), \quad \|\hat{a}\|_\infty = r(a) \leq \|a\| \quad (a \in A). \quad (1)$$

The kernel of Γ is the *radical* $\text{rad}A$, which is the intersection of all maximal ideals, and A is *semisimple* when $\text{rad}A = \{0\}$, that is, when Γ is injective [2].

If A is commutative but has no identity, characters are still continuous, with $\|\varphi\| \leq 1$, but $\Delta(A)$ is only

locally compact in the weak* topology, the Gelfand transforms belong to $C_0(\Delta(A))$, and $\sigma(a) = \hat{a}(\Delta(A)) \cup \{0\}$. The unitisation $A^\# = A \oplus \mathbb{C}e$ has $\Delta(A^\#) = \Delta(A) \cup \{\varphi_\infty\}$, where $\varphi_\infty(a + \lambda e) = \lambda$, and this is the one-point compactification of $\Delta(A)$ [2]. In what follows the unital case is the default; non-unital algebras are treated through their unitisation or their multiplier algebra.

Example 2.1. Let K be a compact Hausdorff space and $A = C(K)$ with the supremum norm. For $x \in K$ the evaluation $\tau_x(f) = f(x)$ is a character with kernel $\{f \in C(K): f(x) = 0\}$, and the map $x \mapsto \tau_x$ is a homeomorphism of K onto $\Delta(A)$. Under this identification $\hat{f} = f$, so the Gelfand map is the identity and $C(K)$ is semisimple.

Example 2.2. Let $A(\mathbb{D})$ be the *disc algebra* of functions continuous on the closed unit disc $\overline{\mathbb{D}}$ and holomorphic in $\mathbb{D}$, with the supremum norm. Every character is evaluation at a point of $\overline{\mathbb{D}}$, and $\Delta(A(\mathbb{D}))$ is homeomorphic to $\overline{\mathbb{D}}$ [2]. Thus $\Delta(A)$ can be strictly larger than the set on which the functions attain their norms, a point taken up in Section III.

Example 2.3. Let G be a locally compact abelian group and $A = L^1(G)$ with convolution. Then A is commutative and semisimple, and $\Delta(A)$ is homeomorphic to the dual group $\hat{G}$, with the Gelfand transform coinciding with the Fourier transform [2]. The algebra $L^1(G)$ has an identity if and only if G is discrete; accordingly $\hat{G}$ is compact only in that case, and in general it is merely locally compact. The natural unital algebra containing $L^1(G)$ as an ideal is the measure algebra $M(G)$, which is the multiplier algebra of $L^1(G)$; this is the setting of the Bochner–Schoenberg–Eberlein theorem discussed in Section V.

Example 2.4. Let $A = \{\alpha e + \beta x: \alpha, \beta \in \mathbb{C}\}$ with $x^2 = 0$ and norm $\|\alpha e + \beta x\| = |\alpha| + |\beta|$. Since $\|(\alpha e + \beta x)(\gamma e + \delta x)\| = |\alpha\gamma| + |\alpha\delta + \beta\gamma| \leq (|\alpha| + |\beta|)(|\gamma| + |\delta|)$, A is a two-dimensional commutative unital Banach algebra. Its only character is $\varphi(\alpha e + \beta x) = \alpha$, so $\Delta(A)$ is a single point, $\text{rad}A = \mathbb{C}x$, and $\hat{x} = 0$. The Gelfand transform therefore does not see the element x at all. We return to this algebra in Example 2.13.

B. Numerical ranges

Let A be a unital Banach algebra, not necessarily commutative. The *state space* of A is

$$S(A) = \{f \in A^*: \|f\| = f(e) = 1\},$$

which is a nonempty, convex, weak* compact subset of A^* .

Definition 2.5. The *algebra numerical range* of $a \in A$ is $V(A, a) = \{f(a): f \in S(A)\}$, and the *numerical radius* of a is $v(a) = \max\{|\lambda|: \lambda \in V(A, a)\}$.

As the image of a weak* compact convex set under the weak* continuous affine map $f \mapsto f(a)$, the set $V(A, a)$ is a compact convex subset of $\mathbb{C}$ [7].

Definition 2.6. Let X be a Banach space with dual X^* and unit sphere S_X , and let $L(X)$ be the algebra of bounded operators on X . The *spatial numerical range* of $T \in L(X)$ is

$$W(T) = \{x^*(Tx): x \in S_X, x^* \in S_{X^*}, x^*(x) = 1\}.$$

In a general Banach space $W(T)$ need not be convex. The two definitions are linked by the following theorem.

Theorem 2.7 (Williams [10]; see also [7]). For every Banach space X and every $T \in L(X)$, $V(L(X), T) = \overline{\text{co}} W(T)$, the closed convex hull of $W(T)$.

When $X = H$ is a Hilbert space, the only functional x^* with $\|x^*\| = x^*(x) = 1$ is $y \mapsto \langle y, x \rangle$, so $W(T) = \{\langle Tx, x \rangle: \|x\| = 1\}$. This set is convex by the Toeplitz–Hausdorff theorem [4, 5], and Theorem 2.7 reduces to $V(L(H), T) = \overline{W(T)}$. The set $W(T)$ is always closed when H is finite dimensional, but need not be closed otherwise; see Section V.

C. The spectrum and the numerical radius

The first basic fact is the inclusion of the spectrum in the numerical range.

Theorem 2.8 ([7]). For every element a of a unital Banach algebra A ,

$$\text{co}\sigma(a) \subseteq V(A, a),$$

and hence $r(a) \leq v(a) \leq \|a\|$.

When A is commutative the inclusion $\sigma(a) \subseteq V(A, a)$ is immediate from (1): each character φ satisfies $\|\varphi\| = \varphi(e) = 1$, so $\Delta(A) \subseteq S(A)$ and $\sigma(a) =$

$\hat{\sigma}(\Delta(A)) \subseteq V(A, a)$. Convexity of $V(A, a)$ then gives $\text{co}\sigma(a) \subseteq V(A, a)$. This is the elementary link on which the rest of the paper builds: the characters form a distinguished subset of the state space, and the numerical range is obtained by evaluating all states rather than characters only.

The upper bound $v(a) \leq \|a\|$ is trivial. A reverse inequality holds with a constant that depends on the setting.

Theorem 2.9. Let A be a unital Banach algebra. Then

$$1/e \|a\| \leq v(a) \leq \|a\| \quad (a \in A), \quad (2)$$

where $e = \exp(1)$. The inequality is due to Bohnenblust and Karlin [11], and the constant $1/e$ cannot be improved in general [12]; see [7].

Theorem 2.10. Let H be a complex Hilbert space and $T \in L(H)$, and write $w(T) = \sup\{|\lambda|: \lambda \in W(T)\}$. Then

$$1/2 \|T\| \leq w(T) \leq \|T\|, \quad (3)$$

with equality on the left for $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and on the right for every normal T [13].

Remark 2.11. The constants in (2) and (3) must not be interchanged. The constant $1/2$ is a Hilbert space phenomenon that relies on the polarisation identity; for a general unital Banach algebra only $1/e$ is available. Both inequalities show that v is a norm on A equivalent to the given norm. This uses the fact that the scalars are complex: over the real field a nonzero element can have numerical range $\{0\}$, as a rotation of the real plane shows.

Definition 2.12. An element h of a unital Banach algebra A is *Hermitian* if $V(A, h) \subseteq \mathbb{R}$. Equivalently, $\|\exp(ith)\| = 1$ for all $t \in \mathbb{R}$ [7].

Example 2.13. Return to the algebra A of Example 2.4. Its dual is identified with $\mathbb{C}^2$ under the pairing $f(ae + \beta x) = \alpha f(e) + \beta f(x)$, with norm $\|f\| = \max\{|f(e)|, |f(x)|\}$. Hence $f \in S(A)$ if and only if $f(e) = 1$ and $|f(x)| \leq 1$, and therefore

$$V(A, x) = \{\lambda \in \mathbb{C}: |\lambda| \leq 1\},$$

$$v(x) = \|x\| = 1, \quad \sigma(x) = \{0\}.$$

Thus a radical element, whose Gelfand transform vanishes identically, can have a numerical range as large as its norm permits. In particular x is not

Hermitian although $\hat{x}$ is real valued, and $V(A, x) \neq \cos(x)$. The same phenomenon occurs on Hilbert space: for the nilpotent matrix $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ one has $\sigma(T) = \{0\}$ while $W(T)$ is the closed disc of radius $1/2$ centred at 0 [9]. These examples show that the numerical range carries information that the maximal ideal space cannot detect, and they will serve as test cases for the statements in Sections IV and V.

III. THE GELFAND REPRESENTATION AND BOUNDARIES

D. The Gelfand representation

Throughout this section A is a commutative unital Banach algebra. The following theorem collects the basic properties of the Gelfand map; see [1] and [2].

Theorem 3.1 (Gelfand). The Gelfand map $\Gamma: A \rightarrow C(\Delta(A))$, $a \mapsto \hat{a}$, is a unital algebra homomorphism with $\|\hat{a}\|_\infty = r(a) \leq \|a\|$. Its range $\hat{A}$ contains the constants and separates the points of $\Delta(A)$, and $\ker \Gamma = \text{rad} A$. Moreover:

1. a is invertible in A if and only if $\hat{a}$ has no zero on $\Delta(A)$;
2. Γ is injective if and only if A is semisimple;
3. Γ is isometric if and only if $\|a^2\| = \|a\|^2$ for every $a \in A$.

Statement (3) follows from the spectral radius formula $r(a) = \lim_n \|a^n\|^{1/n}$: if $\|a^2\| = \|a\|^2$ for all a , then $\|a^{2^k}\|^{2^{-k}} = \|a\|$ for all k , so $r(a) = \|a\|$. An algebra with this property is, through Γ , the same thing as a *uniform algebra*, that is, a closed subalgebra of $C(X)$, for some compact Hausdorff space X , which contains the constants and separates the points of X , normed by the supremum norm. If $A \subseteq C(X)$ is a uniform algebra, then each point $x \in X$ defines the character $f \mapsto f(x)$, and X is homeomorphic to a closed subset of $\Delta(A)$; Example 2.2 shows that this subset can be proper.

The range $\hat{A}$ is dense in $C(\Delta(A))$ only in special cases. The extreme case is described by the commutative Gelfand–Naimark theorem: if A is a commutative unital C^* -algebra, then Γ is an isometric $*$ -isomorphism of A onto $C(\Delta(A))$ [2]. For the disc algebra, by contrast, $\hat{A} = A(\mathbb{D})$ is a proper closed subalgebra of $C(\overline{\mathbb{D}})$.

E. The Šilov boundary

Definition 3.2. A closed subset $F \subseteq \Delta(A)$ is a *boundary* for A if $\max_{\varphi \in F} |\hat{a}(\varphi)| = \|\hat{a}\|_\infty$ for every $a \in A$.

Theorem 3.3 (Šilov; see [2]). The intersection of all boundaries for A is itself a boundary. It is therefore the smallest boundary, called the *Šilov boundary* and denoted ∂A .

For $A = C(K)$ every point of K is a peak point (Urysohn's lemma), so $\partial A = K = \Delta(A)$. For the disc algebra the maximum modulus principle gives $\partial A(\mathbb{D}) = \mathbb{T}$, the unit circle, which is a proper subset of $\Delta(A(\mathbb{D})) = \overline{\mathbb{D}}$.

The Šilov boundary has an extension property that distinguishes its points from the remaining characters.

Theorem 3.4 (Šilov; see [2]). Let B be a commutative unital Banach algebra and A a closed subalgebra of B containing the identity of B . Then every character in ∂A is the restriction to A of a character of B .

For the pair $A(\mathbb{D}) \subseteq C(\mathbb{T})$, where $A(\mathbb{D})$ is identified with its boundary values, every character of $C(\mathbb{T})$ is evaluation at a point of $\mathbb{T}$. Consequently the characters of $A(\mathbb{D})$ corresponding to points of $\partial A(\mathbb{D}) = \mathbb{T}$ extend to $C(\mathbb{T})$, while evaluation at a point of the open disc does not extend. Theorem 3.4 cannot therefore be strengthened to arbitrary characters.

F. Representing measures and the Choquet boundary

Let $A \subseteq C(X)$ be a uniform algebra. A linear functional f on A with $\|f\| = f(1) = 1$ extends, by the Hahn–Banach theorem, to a functional on $C(X)$ with the same two properties, and such a functional is a positive measure by the Riesz representation theorem. Hence every state of A , and in particular every character, has a *representing measure*: a regular Borel probability measure μ on X with

$$f(g) = \int_X g \, d\mu \quad (g \in A).$$

Representing measures need not be unique. For the disc algebra, evaluation at $z \in \mathbb{D}$ is represented by the Poisson measure of z on $\mathbb{T}$, and also by normalised arc

length on every circle about z contained in $\overline{\mathbb{D}}$ when $X = \overline{\mathbb{D}}$.

Definition 3.5. The *Choquet boundary* $\text{Ch}(A)$ of a uniform algebra $A \subseteq C(X)$ is the set of points $x \in X$ for which the point mass δ_x is the only representing measure of the character $g \mapsto g(x)$. A point $x \in X$ is a *peak point* if there is $g \in A$ with $g(x) = 1$ and $|g(y)| < 1$ for all $y \neq x$.

Every peak point lies in the Choquet boundary. The relation between these sets and the Šilov boundary is as follows.

Theorem 3.6 (Bishop [14]; Bishop and de Leeuw [15]). Let $A \subseteq C(X)$ be a uniform algebra.

1. The Choquet boundary is a boundary in the sense that every $|g|$, $g \in A$, attains its maximum on $\text{Ch}(A)$, and $\overline{\text{Ch}(A)} = \partial A$.
2. If X is metrisable, then $\text{Ch}(A)$ coincides with the set of peak points of A .

Without metrisability peak points can fail to exist, so the Choquet boundary, and not the set of peak points, is the correct minimal object in general; see [16, 17] for a full account. The Choquet boundary also has a description in terms of the state space. For a uniform algebra $A \subseteq C(X)$ the extreme points of $S(A)$ are precisely the evaluations at points of $\text{Ch}(A)$ [16]. This is the point at which boundary theory meets numerical range theory, since $V(A, g)$ is the image of $S(A)$ under evaluation at g and a compact convex set is the closed convex hull of its extreme points.

IV. THE NUMERICAL RANGE AND THE CHARACTER SPACE

G. Uniform algebras

For uniform algebras the numerical range is completely determined by the boundary.

Theorem 4.1. Let $A \subseteq C(X)$ be a uniform algebra and $g \in A$. Then

$$V(A, g) = \text{co}g(X) = \text{co}\hat{g}(\partial A) = \text{co}\sigma(g),$$

and consequently $v(g) = r(g) = \|g\|_\infty$.

Proof. By the discussion preceding Definition 3.5, the states of A are exactly the restrictions to A of probability measures on X . Hence $V(A, g) =$

$\{\int_X g \, d\mu\}$, which is the closed convex hull of $g(X)$. Since $g(X)$ is compact in $\mathbb{C}$, its convex hull is already compact (Carathéodory), so $V(A, g) = \text{co}g(X)$.

Next, for $\theta \in \mathbb{R}$ the element $\exp(e^{i\theta}g)$ lies in A and has modulus $\exp \text{Re}(e^{i\theta}g)$. Its maximum modulus is attained on ∂A , so

$$\max_X \text{Re}(e^{i\theta}g) = \max_{\partial A} \text{Re}(e^{i\theta}\hat{g}) \quad (\theta \in \mathbb{R}).$$

Two compact convex subsets of $\mathbb{C}$ with the same support function coincide, so $\text{co}g(X) = \text{co}\hat{g}(\partial A)$, where we use that X , being a closed boundary, contains ∂A under the embedding of X in $\Delta(A)$. Finally $g(X) \subseteq \sigma(g) \subseteq V(A, g)$ by (1) and Theorem 2.8, and taking convex hulls gives $\text{co}\sigma(g) = V(A, g)$. The last assertion follows because the maximum modulus of a point of $\text{co}g(X)$ is attained on $g(X)$. ▀

Thus in a uniform algebra the numerical range is the convex hull of the spectrum, and it is already determined by the values of $\hat{g}$ on the Šilov boundary. The next result shows that, conversely, equality of the numerical radius and the spectral radius forces an algebra to be a uniform algebra up to an equivalent norm.

Proposition 4.2. Let A be a commutative unital Banach algebra such that $v(a) = r(a)$ for every $a \in A$. Then A is semisimple and

$$\|\hat{a}\|_\infty \leq \|a\| \leq e \|\hat{a}\|_\infty \quad (a \in A),$$

so that Γ is a topological isomorphism of A onto the uniform algebra $\hat{A} \subseteq C(\Delta(A))$.

Proof. By Theorem 2.9 and (1), $\|a\| \leq e v(a) = e r(a) = e \|\hat{a}\|_\infty$. Hence $\hat{a} = 0$ implies $a = 0$, and Γ is bounded below, so its range is closed. ▀

Remark 4.3. The converse of Proposition 4.2 fails. Replacing the supremum norm of a uniform algebra by an equivalent algebra norm changes the state space and can enlarge the numerical range beyond $\text{co}\sigma(a)$. Equality $V(A, a) = \text{co}\sigma(a)$ for all a is therefore a property of the norm and not only of the algebraic or topological structure. The radical algebra of Example 2.13 shows the failure in the most extreme form.

H. What the numerical range sees that the Gelfand transform does not

The Gelfand transform annihilates the radical. The numerical range does not.

Corollary 4.4. If a is a nonzero element of a unital Banach algebra, then $v(a) \geq \|a\|/e > 0$. In particular, if A is commutative and $0 \neq a \in \text{rad}A$, then $\hat{a} = 0$ and $\sigma(a) = \{0\}$, but $V(A, a) \neq \{0\}$.

This is immediate from Theorem 2.9. Quantitative information on nilpotent elements is available on Hilbert space.

Theorem 4.5 (Haagerup and de la Harpe [18]). If $T \in L(H)$ satisfies $T^n = 0$ for some $n \geq 2$, then $w(T) \leq \|T\| \cos \frac{\pi}{n+1}$, and the bound is attained.

For $n = 2$ the bound is $1/2 \|T\|$, which agrees with the 2×2 nilpotent matrix of Example 2.13 and with the left-hand case of (3).

1. Hermitian elements

Recall from Definition 2.12 that $h \in A$ is Hermitian when $V(A, h) \subseteq \mathbb{R}$. The key metric fact about Hermitian elements is due to Sinclair.

Theorem 4.6 (Sinclair [19]). If h is a Hermitian element of a unital Banach algebra, then $\|h\| = r(h) = v(h)$.

Combining Theorems 2.8 and 4.6 with (1) gives the correct relation between Hermitian elements and the maximal ideal space.

Corollary 4.7. Let A be a commutative unital Banach algebra and $h \in A$ Hermitian. Then $\hat{h}$ is real valued on $\Delta(A)$ and $\|h\| = \|\hat{h}\|_\infty$. In particular the Hermitian elements of A form a real subspace on which Γ is isometric.

The converse of Corollary 4.7 is false in general: in Example 2.13 the element x has $\hat{x} = 0$, which is real valued, but $V(A, x)$ is the closed unit disc. For uniform algebras, however, Theorem 4.1 yields a complete characterisation in terms of the Šilov boundary.

Corollary 4.8. Let $A \subseteq C(X)$ be a uniform algebra and $g \in A$. The following are equivalent: (i) g is

Hermitian; (ii) g is real valued on X ; (iii) $\hat{g}$ is real valued on ∂A .

Proof. By Theorem 4.1, $V(A, g) = \text{cog}(X) = \text{co}\hat{g}(\partial A)$, and a convex hull lies in $\mathbb{R}$ exactly when the set does. $\square$

In $C(K)$ the Hermitian elements are therefore the real-valued functions, whereas in the disc algebra a function that is real on $\mathbb{T}$ has harmonic imaginary part vanishing on $\mathbb{T}$, hence identically zero, and so is a real constant. The Hermitian elements of $A(\mathbb{D})$ are thus only the real multiples of the identity.

J. The Vidav–Palmer theorem

The real-valued functions span $C(K)$ over $\mathbb{C}$, and this property characterises C^* -algebras among all unital Banach algebras. Write $H(A)$ for the set of Hermitian elements of A .

Theorem 4.9 (Vidav [20], Palmer [21]). Let A be a unital Banach algebra such that $A = H(A) + iH(A)$. Then each $a \in A$ has a unique decomposition $a = h + ik$ with $h, k \in H(A)$, the map $a^* = h - ik$ is an involution, and A with its original norm and this involution is a C^* -algebra.

Vidav proved the theorem under an additional hypothesis on the squares of Hermitian elements, and Palmer showed that this hypothesis is superfluous; see [7, 8] for the complete proof. In the commutative case, Theorem 4.9 combined with the commutative Gelfand–Naimark theorem gives the following numerical range characterisation of the algebras $C(K)$.

Corollary 4.10. A commutative unital Banach algebra A satisfies $A = H(A) + iH(A)$ if and only if Γ is an isometric isomorphism of A onto $C(\Delta(A))$.

Proof. If $A = H(A) + iH(A)$, then A is a commutative C^* -algebra by Theorem 4.9, and the Gelfand–Naimark theorem applies. Conversely, in $C(K)$ the Hermitian elements are the real-valued functions by Corollary 4.8, and every function is the sum of its real part and i times its imaginary part. $\square$

Corollary 4.10 and Corollary 4.8 together locate the uniform algebras between the two extremes: $C(K)$, where $H(A)$ spans everything, and algebras such as $A(\mathbb{D})$, where $H(A) = \mathbb{R}e$. In both cases the numerical

range is governed by the Šilov boundary through Theorem 4.1.

V. DEVELOPMENTS SINCE 1990

This section reviews six lines of work that connect maximal ideals, states and numerical ranges. Each result is quoted from the source indicated; results that are elementary consequences of Sections II to IV are given with proofs.

K. States generated by characters

Section IV showed that $V(A, a) = \text{co}\sigma(a)$ for every element of a uniform algebra but not for every commutative Banach algebra. The following observation isolates the property of the state space that is responsible.

Proposition 5.1. Let A be a commutative unital Banach algebra and let K be the weak* closed convex hull of $\Delta(A)$ in A^* . Then $K \subseteq S(A)$, and the following are equivalent:

1. $V(A, a) = \text{co}\sigma(a)$ for every $a \in A$;
2. $S(A) = K$.

Proof. Since $\Delta(A) \subseteq S(A)$ and $S(A)$ is weak* closed and convex, $K \subseteq S(A)$. For every a , the set $\{f(a): f \in K\}$ is the closed convex hull of $\hat{a}(\Delta(A)) = \sigma(a)$, which equals $\text{co}\sigma(a)$ because $\sigma(a)$ is compact. Hence (2) implies (1). Conversely, suppose $f_0 \in S(A) \setminus K$. By the Hahn–Banach separation theorem in the weak* topology, whose continuous linear functionals are the evaluations at elements of A , there is $a \in A$ with $\text{Re} f_0(a) > \sup_{f \in K} \text{Re} f(a)$. Then $f_0(a) \in V(A, a) \setminus \text{co}\sigma(a)$, so (1) fails. ▀

By Theorem 4.1, every uniform algebra satisfies the conditions of Proposition 5.1, and by Proposition 4.2 every algebra that satisfies them is a uniform algebra up to an equivalent norm. The radical algebra of Example 2.13 is the simplest algebra in which K is a proper subset of $S(A)$: there $K = \{\varphi\}$ is a single point while $S(A)$ is a disc.

L. The Bochner–Schoenberg–Eberlein property

Let A be a commutative semisimple Banach algebra, not necessarily unital, and let $M(A)$ denote its multiplier algebra, the algebra of bounded operators T on A with $T(ab) = aT(b)$ for all a, b . Every

multiplier has a Gelfand transform $\hat{T} \in C_b(\Delta(A))$, determined by $\widehat{Ta} = \hat{T}\hat{a}$. A bounded continuous function σ on $\Delta(A)$ is a *BSE-function* if there is $\beta > 0$ such that

$$\left| \sum_{j=1}^n c_j \sigma(\varphi_j) \right| \leq \beta \left\| \sum_{j=1}^n c_j \varphi_j \right\|_{A^*} \quad (4)$$

for all n , all $c_1, \dots, c_n \in \mathbb{C}$ and all $\varphi_1, \dots, \varphi_n \in \Delta(A)$. The infimum of such β is the BSE-norm of σ , and the set of BSE-functions is denoted $C_{\text{BSE}}(\Delta(A))$.

Definition 5.2 (Takahasi and Hatori [22]). A is a *BSE-algebra* if $C_{\text{BSE}}(\Delta(A)) = \widehat{M(A)}$.

The terminology comes from the classical theorem of Bochner, Schoenberg and Eberlein, which states that for a locally compact abelian group G a bounded continuous function on $\hat{G}$ is the Fourier–Stieltjes transform of a bounded measure on G precisely when it satisfies an inequality of the form (4). Since $M(L^1(G)) = M(G)$, this says exactly that $L^1(G)$ is a BSE-algebra. Takahasi and Hatori [22] showed that $C_{\text{BSE}}(\Delta(A))$ is a commutative semisimple Banach algebra under the BSE-norm and initiated the general theory; the systematic study was carried out by Kaniuth and Ülger [23], and related extension problems for homomorphisms into multiplier algebras were treated by Kaniuth, Lau and Ülger [24]. Among the results of [23] is the following.

Theorem 5.3 (Kaniuth and Ülger [23]). Let A be a commutative semisimple Banach algebra which is an ideal in its second dual. Then the following are equivalent: (i) A is a BSE-algebra; (ii) A has a bounded weak approximate identity; (iii) A has a bounded approximate identity.

The BSE property has since been determined for many concrete algebras, among them $L^1(\mathbb{R}^+)$ [25] and several classes of Segal algebras [26]. It is important to be clear about what the property is. Condition (4) compares a function on the character space with the dual norm of linear combinations of characters, and so it concerns the subspace of A^* spanned by $\Delta(A)$ and the geometry that A^* induces on it. It is not a statement about numerical ranges, and it does not imply the equality $V(A, a) = \text{co}\sigma(a)$ of Proposition 5.1. For

example, $\ell^1(\mathbb{Z}) = L^1(\mathbb{Z})$ is a unital BSE-algebra by the classical theorem. Its states are the sequences $f \in \ell^\infty(\mathbb{Z})$ with $f_0 = 1$ and $\|f\|_\infty = 1$, so for $a = \delta_1 + \delta_{-1}$ the choice $f_1 = f_{-1} = i$ gives $2i \in V(A, a)$, whereas $\hat{a}(\theta) = 2\cos\theta$ and $\cos\sigma(a) = [-2, 2]$. The two notions nevertheless share a common setting, since both are governed by the position of $\Delta(A)$ inside the unit ball of A^* ; a precise link between them is not known and is formulated as a problem in Section VII.

M. Numerical radius inequalities on Hilbert space

On a Hilbert space the inequalities (3) have been refined in many directions. Two refinements due to Kittaneh are now standard.

Theorem 5.4 (Kittaneh [27, 28]). For every $T \in L(H)$,

$$w(T) \leq 1/2 (\|T\| + \|T^2\|^{1/2}),$$

$$1/4 \|T^*T + TT^*\| \leq w(T)^2 \leq 1/2 \|T^*T + TT^*\|.$$

Corollary 5.5. If $T^2 = 0$, then $w(T) = 1/2 \|T\|$.

Proof. The first inequality of Theorem 5.4 gives $w(T) \leq \frac{1}{2} \|T\|$, and (3) gives the reverse inequality. $\square$

Corollary 5.5 is the case $n = 2$ of Theorem 4.5. Many further upper and lower bounds have been obtained since, for instance by Bhunia and Paul [29]; the monograph of Bhunia, Dragomir, Moslehian and Paul [30] gives a systematic account, and extensions to operators on Hilbert C^* -modules are developed by Moghaddam and Mirmostafaei [31].

N. Closedness of the numerical range of compact operators

The numerical range of an operator on an infinite-dimensional Hilbert space need not be closed, even for compact operators. Halmos [13] gave the example

$$T = \text{diag}(1, 1/2, 1/3, \dots) \text{ on } \ell^2, \quad W(T) = (0, 1],$$

of a compact, positive, and in particular normal operator whose numerical range is not closed. The question of closedness is answered completely by the following theorem.

Theorem 5.6 (de Barra, Giles and Sims [32]). Let T be a compact operator on an infinite-dimensional Hilbert space H .

1. If $0 \in W(T)$, then $W(T)$ is closed.
2. If $0 \notin W(T)$, then 0 is an extreme point of $\overline{W(T)}$, and $\overline{W(T)} \setminus W(T)$ consists at most of line segments

in the boundary of $W(T)$ that contain 0 and no other extreme point of $\overline{W(T)}$.

Corollary 5.7. Let T be compact on an infinite-dimensional Hilbert space. Then $W(T)$ is closed if and only if $0 \in W(T)$. If H is non-separable, $W(T)$ is always closed.

Proof. Since H is infinite dimensional and T is compact, $0 \in \sigma(T) \subseteq \overline{W(T)}$. If $0 \notin W(T)$, then $0 \in \overline{W(T)} \setminus W(T)$ and $W(T)$ is not closed; the converse is part (1) of Theorem 5.6. If H is non-separable, the range of T is separable, so T has a nonzero kernel and 0 is an eigenvalue, whence $0 \in W(T)$. $\square$

The Halmos example satisfies $0 \notin W(T)$, in accordance with Corollary 5.7. The example also shows that normality does not help: for a normal operator one has $\overline{W(T)} = \cos\sigma(T)$ [9], but $W(T)$ itself need not be closed. On a finite-dimensional space, by contrast, $W(T)$ is always compact, being the continuous image of the unit sphere.

O. M-ideals, quotients and the essential numerical range

A closed subspace J of a Banach space X is an *M-ideal* if its annihilator $J^\perp \subseteq X^*$ is the range of an L -projection, that is, of a projection P with $\|f\| = \|Pf\| + \|f - Pf\|$ for all $f \in X^*$ [33]. In a C^* -algebra the M -ideals are exactly the closed two-sided ideals [34, 33], and the ideal $K(H)$ of compact operators is an M -ideal in $L(H)$ [33]. The study of M -ideals in general Banach algebras, and their relation to the algebraic ideal structure, goes back to Smith and Ward [34]; the monograph [33] gives a comprehensive account, including the M -ideals of uniform algebras.

Quotients by ideals interact with numerical ranges in a simple way.

Proposition 5.8. Let A be a unital Banach algebra, J a proper closed two-sided ideal and $q: A \rightarrow A/J$ the quotient map. Then for every $a \in A$,

$$V(A/J, a + J) \subseteq \bigcap_{j \in J} V(A, a + j).$$

Proof. The quotient A/J is a unital Banach algebra with identity $e + J$, and $\|e + J\| = 1$ because $1 = r(e + J) \leq \|e + J\| \leq \|e\| = 1$. If g is a state of A/J ,

then $f = g \circ q$ satisfies $\|f\| \leq 1$ and $f(e) = 1$, so $f \in S(A)$. For $j \in J$ we have $g(a + j) = f(a + j) \in V(A, a + j)$. *

When $A = L(H)$ and $J = K(H)$, the set on the left is the numerical range of $T + K(H)$ in the Calkin algebra. It coincides with the *essential numerical range*

$$W_e(T) = \bigcap_{K \in K(H)} \overline{W(T + K)},$$

which contains the essential spectrum and plays for Fredholm theory the role that $\overline{W(T)}$ plays for the spectrum [9]. The question of when equality holds in Proposition 5.8 for other pairs (A, J) , and whether the M-ideal property of J is the decisive condition, is taken up in Section VII.

P. Unbounded operators

For a densely defined operator T with domain $\mathcal{D}(T)$ in a Hilbert space, the numerical range $W(T) = \{Tx, x : x \in \mathcal{D}(T), \|x\| = 1\}$ is still convex by the Toeplitz–Hausdorff argument, but it may be unbounded, and its relation to the spectrum is weaker than in the bounded case. Bögli, Marletta and Tretter [35] developed the essential numerical range for unbounded operators, established its basic properties and its relation to the essential spectrum, and applied it to enclose the spurious eigenvalues (spectral pollution) produced by projection methods. This extends to the unbounded setting the role that $W_e(T)$ plays in the bounded case.

Q. Non-commutative boundaries

The Šilov and Choquet boundaries of Section III have non-commutative analogues, introduced by Arveson [36] for operator systems, that is, self-adjoint unital subspaces S of a C^* -algebra. A *boundary representation* of S is an irreducible representation of the generated C^* -algebra whose restriction to S has a unique unital completely positive extension; this is the analogue of a point of the Choquet boundary, where the character has a unique representing measure. The analogue of the Šilov boundary is the C^* -envelope, the smallest C^* -algebra generated by a completely isometric copy of S . Hamana [37] proved that the C^* -envelope always exists, using injective envelopes. Arveson [38] proved that a separable operator system has sufficiently many boundary representations to

determine its C^* -envelope, and Davidson and Kennedy [39] removed the separability assumption, so that the C^* -envelope is always determined by the non-commutative Choquet boundary. Kleski [40] related boundary representations to pure unital completely positive maps.

For a uniform algebra $A \subseteq C(X)$ the theory reduces to Section III: the C^* -envelope of $A + \overline{A}$ is $C(\partial A)$, and the boundary representations correspond to the points of the Choquet boundary. The connection with numerical ranges passes through state spaces: for $T \in L(H)$, the operator system S_T spanned by $1, T$ and T^* has states whose values at T fill out $\overline{W(T)}$, so the non-commutative boundary of S_T is a natural refinement of the extreme points of the numerical range.

VI. APPLICATIONS

R. Exponential bounds and dissipative elements

The numerical range controls the growth of exponentials in any unital Banach algebra. For $a \in A$ write $\omega(a) = \max\{\operatorname{Re} \lambda : \lambda \in V(A, a)\}$.

Theorem 6.1 (Lumer [6]; see [7]). Let A be a unital Banach algebra and $a \in A$. Then

$$\omega(a) = \lim_{t \rightarrow 0^+} \frac{\|e + ta\| - 1}{t},$$

$$\|\exp(ta)\| \leq \exp(t\omega(a)) \quad (t \geq 0).$$

An element with $\omega(a) \leq 0$ is called *dissipative*; by Theorem 6.1 it generates a contraction semigroup $t \mapsto \exp(ta)$, and this is the algebraic form of the Lumer–Phillips theorem. The same estimate applied to ih and $-ih$ gives the equivalence in Definition 2.12: h is Hermitian precisely when $\|\exp(ith)\| = 1$ for all real t .

In a commutative algebra the spectral bound $\max \operatorname{Re} \sigma(a) = \max \operatorname{Re} \hat{a}(\Delta(A))$ governs the asymptotic growth of $\|\exp(ta)\|$, since $r(\exp(ta)) = \exp(t \max \operatorname{Re} \sigma(a))$. It does not bound the norm for small t . For the element x of Example 2.13 one has $\max \operatorname{Re} \sigma(x) = 0$ but $\omega(x) = 1$, and indeed $\|\exp(tx)\| = \|e + tx\| = 1 + t$. The numerical range therefore measures transient growth that is invisible on the maximal ideal space. For

uniform algebras the two bounds coincide by Theorem 4.1.

S. Spectral sets and the Crouzeix conjecture

Let $T \in L(H)$ and let $\Omega \subseteq \mathbb{C}$ be a compact set containing $\sigma(T)$. The set Ω is a K -spectral set for T if $\|p(T)\| \leq K \max_{z \in \Omega} |p(z)|$ for every polynomial p . By von Neumann's inequality the closed unit disc is a 1-spectral set for every contraction. The question of whether the numerical range is a K -spectral set with K independent of T and of H has driven much recent work.

Two cases are elementary. If T is normal, then $\|p(T)\| = \max_{\sigma(T)} |p|$ and $\sigma(T) \subseteq \overline{W(T)}$, so $\overline{W(T)}$ is 1-spectral. If $A \subseteq C(X)$ is a uniform algebra and $g \in A$, then $\|p(g)\|_\infty = \max_X |p \circ g| \leq \max_{V(A,g)} |p|$, so $V(A, g)$ is a 1-spectral set in the algebra sense. In both cases the structure of the maximal ideal space (the spectral theorem, or Theorem 4.1) makes the numerical range as small as possible. For non-normal operators this is no longer so, and the following conjecture is open.

Problem 6.2 (Crouzeix's conjecture [41, 42]). For every Hilbert space operator T and every polynomial p ,

$$\|p(T)\| \leq 2 \max_{z \in \overline{W(T)}} |p(z)|.$$

The constant 2 cannot be lowered: for $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $p(z) = z$ one has $\|p(T)\| = 1$ while $\overline{W(T)}$ is the disc of radius $\frac{1}{2}$. The current state of knowledge is summarised in the following theorem.

Theorem 6.3.

1. (Crouzeix [41]) The conjecture holds for 2×2 matrices.
2. (Okubo and Ando [43]) If $\overline{W(T)}$ is a disc, then it is a 2-spectral set for T ; in particular $w(T) \leq 1$ implies $\|p(T)\| \leq 2 \max_{|z| \leq 1} |p(z)|$.
3. (Crouzeix [42]) For every T , $\overline{W(T)}$ is a K -spectral set with $K \leq 11.08$.
4. (Crouzeix and Palencia [44]) For every T , $\overline{W(T)}$ is a $(1 + \sqrt{2})$ -spectral set.

The Crouzeix–Palencia proof is short and rests on an abstract functional-analytic lemma; Ransford and Schwenninger [45] gave a new proof of that lemma

and showed that $1 + \sqrt{2}$ is the best constant obtainable from it, so that a proof of the conjecture will require further information about the operator. A related estimate for the numerical radius is due to Drury: if $w(T) \leq 1$ and f belongs to the disc algebra with $\|f\|_\infty \leq 1$, then $w(f(T)) \leq \frac{5}{4}$ [46].

The proofs of these results use the Hilbert space structure in an essential way. For the algebra numerical range $V(A, a)$ of an element of a general unital Banach algebra, the analogous question is considered in Section VII.

T. Gleason parts

Let $A \subseteq C(X)$ be a uniform algebra. Since characters have norm one, $\|\varphi - \psi\|_{A^*} \leq 2$ for $\varphi, \psi \in \Delta(A)$, and

$$\|\varphi - \psi\|_{A^*} = \sup\{|\hat{a}(\varphi) - \hat{a}(\psi)| : \|a\| \leq 1\}.$$

Gleason showed that the relation $\|\varphi - \psi\| < 2$ is an equivalence relation on $\Delta(A)$; its equivalence classes are the *Gleason parts* of A [17]. The norm distance between characters used here is the same dual norm that appears in the state space $S(A)$ and in the BSE inequality (4), so Gleason parts describe the metric position of $\Delta(A)$ inside the unit ball of A^* from a third point of view.

Two characterisations are standard [17]. First, φ and ψ lie in the same part if and only if they have representing measures μ_φ and μ_ψ that are mutually bounded, $c^{-1}\mu_\psi \leq \mu_\varphi \leq c\mu_\psi$ for some $c \geq 1$ (Harnack's inequality in abstract form). Secondly, every point of the Choquet boundary, and in particular every peak point, forms a one-point part, because its only representing measure is a point mass.

For $C(K)$ every part is a single point. For the disc algebra the open disc $\mathbb{D}$ is one part, on which the Gelfand transforms are the holomorphic functions, and each point of $\mathbb{T}$ is a one-point part. It would be natural to expect that a nontrivial part always carries analytic structure in this way, and for certain classes of uniform algebras, such as Dirichlet algebras, this is true [17]. In general it is false. Cole constructed the first uniform algebra with a nontrivial Gleason part but no analytic discs in its maximal ideal space (see [47] and the references there); Izzo and Papathanasiou [47] strengthened Garnett's theorem that every σ -compact completely regular space occurs as a Gleason part of some uniform algebra, obtaining such realisations in

maximal ideal spaces that contain no analytic discs. The relation between one-point parts and point derivations, which measure first-order analytic structure at a character, is investigated by Ghosh and Izzo [48].

The argument through unique representing measures applies only to points of $\text{Ch}(A)$. Since $\partial A = \overline{\text{Ch}(A)}$ can be strictly larger than $\text{Ch}(A)$, the Choquet boundary, and not the Šilov boundary, is the correct setting for the statement that boundary points form one-point parts.

VII. OPEN PROBLEMS AND CONCLUSION

U. Open problems

The following questions arise from the material of Sections IV to VI. Each is stated in terms of the results proved or quoted above.

Problem 7.1. Prove or disprove Crouzeix's conjecture (Problem 6.2). Independently of the general case, determine the optimal constant K for which $\overline{W(T)}$ is a K -spectral set for natural classes of operators, beyond the 2×2 matrices and the operators whose numerical range is a disc covered by Theorem 6.3.

Problem 7.2. Characterise the commutative unital Banach algebras A whose state space is the weak* closed convex hull of $\Delta(A)$, equivalently, by Proposition 5.1, those in which $V(A, a) = \text{co}\sigma(a)$ for every a . Uniform algebras have this property (Theorem 4.1), and every algebra with this property is a uniform algebra up to an equivalent norm (Proposition 4.2). Must such an algebra be isometrically isomorphic to a uniform algebra, that is, must its norm equal $\|\hat{a}\|_\infty$?

Problem 7.3. The BSE inequality (4), the state space $S(A)$ and the Gleason distance $\|\varphi - \psi\|_{A^*}$ are all determined by the position of $\Delta(A)$ in the unit ball of A^* . For a unital commutative semisimple Banach algebra, find a characterisation of the BSE property in terms of the weak* closed convex hull K of $\Delta(A)$ and the state space, or show that the BSE property is independent of the relation between K and $S(A)$. The example of $\ell^1(\mathbb{Z})$ in Section V shows that the BSE property does not force $K = S(A)$.

Problem 7.4. For which proper closed two-sided ideals J of a unital Banach algebra A does equality hold in Proposition 5.8, that is,

$$V(A/J, a + J) = \bigcap_{j \in J} V(A, a + j) \quad (a \in A)?$$

Equality holds for $J = K(H)$ in $A = L(H)$, by the identification of the numerical range in the Calkin algebra with the essential numerical range $W_e(T)$. Does it hold for every M-ideal J , and can it hold for ideals that are not M-ideals?

Problem 7.5. For a uniform algebra A , the Gleason parts, the Choquet boundary and the extreme points of the state space are all described by the dual norm on $\text{span}\Delta(A)$. Find a description of the Gleason parts of A , and in particular of its one-point parts, in terms of the numerical ranges $V(A, a)$, $a \in A$, and the faces of the state space $S(A)$ that these numerical ranges expose.

VIII. CONCLUSION

The maximal ideal space and the numerical range meet in one elementary fact: every character of a unital Banach algebra is a state. From it follow the inclusion $\text{co}\sigma(a) \subseteq V(A, a)$ and the comparison $r(a) \leq v(a) \leq \|a\|$, together with the sharp reverse bounds $\|a\| \leq e v(a)$ in general and $\|T\| \leq 2w(T)$ on Hilbert space. For uniform algebras the connection is exact: the numerical range is the convex hull of the spectrum and is determined by the Šilov boundary, the Hermitian elements are the functions that are real on that boundary, and by the Vidav–Palmer theorem the algebras $\mathcal{C}(K)$ are characterised by the abundance of Hermitian elements. Beyond uniform algebras the numerical range records information that the Gelfand transform discards. It detects every nonzero element of the radical, it bounds the transient growth of exponentials that the spectrum cannot see, and on Hilbert space it supports a functional calculus whose best constant, lying between 2 and $1 + \sqrt{2}$, is the subject of Crouzeix's conjecture. The M-ideal structure of an algebra, the Bochner–Schoenberg–Eberlein property and the theory of Gleason parts all concern the same object, the position of the characters inside the unit ball of the dual space, and the problems above ask how these descriptions fit together.

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