

Odd Exponentiated Half-Logistic Exponentiated Exponential Distribution: Properties, Estimation and Simulation Study

ADEWOYE, KUNLE BAYO^{1*}, AZEEZ, OLASUNKANMI, ISIAKA^{2*}, AZEEZ, OLABISI OMODASOLA³

^{1, 2, 3} Department of Statistics, Federal Polytechnic Offa, Kwara State, Nigeria

Abstract- This paper introduces a new four-parameter lifetime distribution referred to as the Odd Exponentiated Half-Logistic Exponentiated Exponential (OEHL-EE) distribution. The proposed model is obtained by incorporating the Exponentiated Exponential distribution into the Odd Exponentiated Half-Logistic-G generator framework. The additional shape parameters provide increased flexibility for modelling lifetime and reliability data exhibiting skewness, varying kurtosis and different hazard-rate behaviours. Several mathematical and statistical properties of the proposed distribution are derived, including the probability density function, cumulative distribution function, survival function, hazard function, reversed hazard function, cumulative hazard function, quantile function, moments, moment generating function, coefficient of variation, skewness, kurtosis, Rényi entropy and order statistics. Maximum likelihood estimation is employed for estimating the unknown parameters, while asymptotic confidence intervals are obtained from the observed information matrix. A Monte Carlo simulation study is conducted to investigate the performance of the maximum likelihood estimators at different sample sizes. The simulation results indicate that the estimation procedure provides useful parameter estimates and that the magnitude of estimation error generally decreases as the sample size increases. The proposed OEHL-EE distribution therefore provides a flexible alternative for lifetime and reliability modelling.

Keywords: Exponentiated Exponential distribution; Half-Logistic distribution; Odd Exponentiated Half-Logistic-G; lifetime distribution; maximum likelihood estimation; Monte Carlo simulation; reliability analysis.

I. INTRODUCTION

Probability distributions play an important role in statistical modelling because they provide mathematical frameworks for describing random phenomena observed in different fields. In lifetime and reliability analysis, classical distributions such as

the Exponential, Weibull and Gamma distributions have been widely employed. However, real-life lifetime data may exhibit characteristics such as substantial skewness, heavy tails, different levels of kurtosis and non-monotonic hazard-rate behaviour that cannot always be adequately represented by classical models.

One approach to improving the flexibility of a probability distribution is to introduce additional shape parameters through suitable generator mechanisms. Generated families of distributions have consequently received considerable attention because they can retain desirable properties of a baseline distribution while providing additional flexibility in statistical modelling.

The Exponentiated Exponential (EE) distribution is an extension of the classical Exponential distribution designed to provide greater flexibility in modelling lifetime data. The Odd Exponentiated Half-Logistic-G generator provides another framework through which additional shape parameters can be introduced into an existing baseline distribution. The present study combines these two ideas to develop a new lifetime model called the Odd Exponentiated Half-Logistic Exponentiated Exponential (OEHL-EE) distribution.

The proposed model is a four-parameter distribution with parameters α, λ, γ and β . The additional parameters provide flexibility in controlling the shape, scale, skewness and tail behaviour of the resulting distribution.

The thesis from which this manuscript is developed derives several important properties of the OEHL-EE distribution, including ordinary moments, mean, variance, coefficient of variation, skewness, kurtosis,

moment generating function, entropy, reliability and hazard functions, quantile function and order statistics. The objectives of this paper are therefore to:

1. introduce the OEHL-EE distribution;
2. derive its important mathematical and statistical properties;
3. obtain parameter estimates using maximum likelihood estimation;
4. establish asymptotic confidence intervals for the model parameters; and
5. investigate the finite-sample performance of the estimators using Monte Carlo simulation.

The remainder of the paper is organized as follows. Section 2 introduces the OEHL-EE distribution. Section 3 presents its statistical properties. Section 4 discusses parameter estimation. Section 5 presents the simulation study and results. Section 6 provides the concluding remarks.

II. THE OEHL-EE DISTRIBUTION

2.1 Definition: Suppose that T is a lifetime random variable with a certain continuous cumulative distribution function (cdf), $G(x; \xi)$. The odds ratio that an object which follows the lifetime T will die at time x is $\frac{G(x; \xi)}{1-G(x; \xi)}$. If the variability of these odds of death is represented by the random variable X and assumed it follows OEHL-G distributions then according to Afify et al. (2017), the cdf of OEHL-G is defined as

$$F(x; \alpha, \lambda, \xi) = \left[\frac{1 - \exp\left(\frac{-\lambda(G(x; \xi))}{1-G(x; \xi)}\right)}{1 + \exp\left(\frac{-\lambda(G(x; \xi))}{1-G(x; \xi)}\right)} \right]^\alpha \quad (1)$$

and the corresponding probability density function (pdf) of the OEHL-G family of distributions is obtained as

$$f(x; \alpha, \lambda, \xi) = \frac{2\alpha\lambda(g(x; \xi)) \left[\exp\left(\frac{-\lambda(G(x; \xi))}{1-G(x; \xi)}\right) \right] \left[1 - \exp\left(\frac{-\lambda(G(x; \xi))}{1-G(x; \xi)}\right) \right]^{\alpha-1}}{[1-G(x; \xi)]^2 \left[1 + \exp\left(\frac{-\lambda(G(x; \xi))}{1-G(x; \xi)}\right) \right]^{\alpha+1}} \quad (2)$$

where,

$G(x; \xi)$ is the cdf of baseline distribution which depends on a parameter vector ξ ,

$g(x; \xi)$ is the pdf of baseline distribution which depends on a parameter vector ξ ,

α is a shape parameter; $\alpha > 0$,

λ is a shape parameter; $\lambda > 0$,

$x > 0$.

2.2 Odd Exponentiated Half Logistic Exponentiated Exponential (OEHL-EE)

Exponentiated Exponential (EE) Distribution

Definition: A random variable X is said to follow exponentiated exponential (EE)

distribution with parameters β and γ , if its cdf has the following form;

$$G(x) = (1 - \exp(-\beta x))^\gamma \quad (3)$$

The corresponding pdf of the EE distribution is given as

$$g(x) = \gamma\beta(\exp(-\beta x))(1 - \exp(-\beta x))^{\gamma-1} \quad (4)$$

$x > 0; \beta, \gamma > 0$.

The EE distribution in equation (3) has shape parameters γ , and scale parameter β . Note that the shape parameter γ represent the exponential family when it is equaled to 1 (Gupta and Kundu, 2001).

Suppose a random variable X follows an OEHL-EE distribution with parameters $\alpha, \lambda, \gamma, \beta$ and using the cdf and pdf of EE distribution as the baseline distribution then the cdf of OEHL-EE distribution is obtained by substituting equation (3) in equation (1); and it is defined as

$$F(x; \alpha, \lambda, \gamma, \beta) = \left[\frac{1 - \exp\left(\frac{-\lambda(1 - \exp(-\beta x))^\gamma}{1 - (1 - \exp(-\beta x))^\gamma}\right)}{1 + \exp\left(\frac{-\lambda(1 - \exp(-\beta x))^\gamma}{1 - (1 - \exp(-\beta x))^\gamma}\right)} \right]^\alpha \quad (5)$$

The corresponding pdf of OEHL-EE distribution is obtained by differentiating (5) with respect to x or by substituting equations (3) and (4) in equation (2), we have

$$f(x; \alpha, \lambda, \gamma, \beta) = \frac{2\alpha\lambda(\gamma\beta(\exp(-\beta x))(1-\exp(-\beta x))^{\gamma-1}) \left[\exp\left(\frac{-\lambda(1-\exp(-\beta x))^{\gamma}}{1-(1-\exp(-\beta x))^{\gamma}}\right) \right] \left[1-\exp\left(\frac{-\lambda(1-\exp(-\beta x))^{\gamma}}{1-(1-\exp(-\beta x))^{\gamma}}\right) \right]^{\alpha-1}}{\left[1-(1-\exp(-\beta x))^{\gamma} \right]^2 \left[1+\exp\left(\frac{-\lambda(1-\exp(-\beta x))^{\gamma}}{1-(1-\exp(-\beta x))^{\gamma}}\right) \right]^{\alpha+1}} \quad (6)$$

The hazard function is defined as

$$x \in (0, \infty); \alpha, \lambda, \gamma, \beta > 0$$

where, α, λ , and γ are the shape parameters; β is the scale parameter.

2.3 Hazard Function

The hazard function is also known as the failure or risk function and is the probability that a system or an individual or a unit will fail or die for an interval of time.

$$h(x) = \frac{f(x)}{1-F(x)} = \frac{f(x)}{S(x)} \quad (7)$$

where, $F(x)$ and $f(x)$ are the cdf and pdf of a baseline (parent) distribution, respectively.

The hazard function of OEHL-EE distribution is given as

$$h(x) = \frac{2\alpha\lambda(\gamma\beta(\exp(-\beta x))(1-\exp(-\beta x))^{\gamma-1}) \left[\exp\left(\frac{-\lambda(1-\exp(-\beta x))^{\gamma}}{1-(1-\exp(-\beta x))^{\gamma}}\right) \right] \left[1-\exp\left(\frac{-\lambda(1-\exp(-\beta x))^{\gamma}}{1-(1-\exp(-\beta x))^{\gamma}}\right) \right]^{\alpha-1}}{\left\{ 1 - \left[\frac{1-\exp\left(\frac{-\lambda(1-\exp(-\beta x))^{\gamma}}{1-(1-\exp(-\beta x))^{\gamma}}\right)}{1+\exp\left(\frac{-\lambda(1-\exp(-\beta x))^{\gamma}}{1-(1-\exp(-\beta x))^{\gamma}}\right)} \right]^{\alpha} \right\} \left\{ \left[1-(1-\exp(-\beta x))^{\gamma} \right]^2 \left[1+\exp\left(\frac{-\lambda(1-\exp(-\beta x))^{\gamma}}{1-(1-\exp(-\beta x))^{\gamma}}\right) \right]^{\alpha+1} \right\}} \quad (8)$$

2.4 Reversed Hazard Function of OEHL-EE Distribution

The reversed hazard function of OEHL-EE distribution is given as

$$r(x) = \frac{2\alpha\lambda(\gamma\beta(\exp(-\beta x))(1-\exp(-\beta x))^{\gamma-1}) \left[\exp\left(\frac{-\lambda(1-\exp(-\beta x))^{\gamma}}{1-(1-\exp(-\beta x))^{\gamma}}\right) \right] \left[1-\exp\left(\frac{-\lambda(1-\exp(-\beta x))^{\gamma}}{1-(1-\exp(-\beta x))^{\gamma}}\right) \right]^{\alpha-1}}{\left\{ \frac{1-\exp\left(\frac{-\lambda(1-\exp(-\beta x))^{\gamma}}{1-(1-\exp(-\beta x))^{\gamma}}\right)}{1+\exp\left(\frac{-\lambda(1-\exp(-\beta x))^{\gamma}}{1-(1-\exp(-\beta x))^{\gamma}}\right)} \right\}^{\alpha} \left\{ \left[1-(1-\exp(-\beta x))^{\gamma} \right]^2 \left[1+\exp\left(\frac{-\lambda(1-\exp(-\beta x))^{\gamma}}{1-(1-\exp(-\beta x))^{\gamma}}\right) \right]^{\alpha+1} \right\}} \quad (9)$$

2.5 Cumulative Hazard Function of OEHL-EE Distribution

The cumulative hazard function of OEHL-EE distribution is given as

$$H(x) = \ln \left[1 - \left\{ \frac{1-\exp\left(\frac{-\lambda(1-\exp(-\beta x))^{\gamma}}{1-(1-\exp(-\beta x))^{\gamma}}\right)}{1+\exp\left(\frac{-\lambda(1-\exp(-\beta x))^{\gamma}}{1-(1-\exp(-\beta x))^{\gamma}}\right)} \right\}^{\alpha} \right] \quad (10)$$

2.6 Survival or Reliability Function of OEHL-EE Distribution

The survival function of OEHL-EE distribution is given as

$$S(x) = 1 - \left\{ \frac{1-\exp\left(\frac{-\lambda(1-\exp(-\beta x))^{\gamma}}{1-(1-\exp(-\beta x))^{\gamma}}\right)}{1+\exp\left(\frac{-\lambda(1-\exp(-\beta x))^{\gamma}}{1-(1-\exp(-\beta x))^{\gamma}}\right)} \right\}^{\alpha} \quad (11)$$

2.7 Odd Function of OEHL-EE Distribution

The odd function of OEHL-EE distribution is given as

$$ODD(x) = \frac{\left[\frac{1-\exp\left(\frac{-\lambda(1-\exp(-\beta x))^{\gamma}}{1-(1-\exp(-\beta x))^{\gamma}}\right)}{1+\exp\left(\frac{-\lambda(1-\exp(-\beta x))^{\gamma}}{1-(1-\exp(-\beta x))^{\gamma}}\right)} \right]^{\alpha}}{1 - \left\{ \frac{1-\exp\left(\frac{-\lambda(1-\exp(-\beta x))^{\gamma}}{1-(1-\exp(-\beta x))^{\gamma}}\right)}{1+\exp\left(\frac{-\lambda(1-\exp(-\beta x))^{\gamma}}{1-(1-\exp(-\beta x))^{\gamma}}\right)} \right\}^{\alpha}} \quad (12)$$

Plots of the density function (9) can be represented through Figure 1. As it seems from Figure 1 that the pdf of OEHL-EE can take different shapes according to different values of α and λ . It can be symmetric, right skewed, unimodal

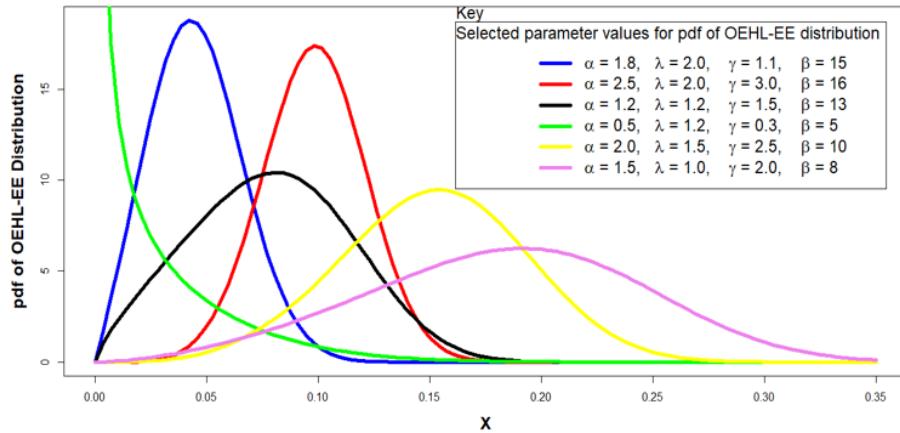


Figure 1: Plot of pdf for some selected parameter values of OEHL-EE distribution

Plots of cumulative distribution function (8) for the OEHL-EE distribution are displayed in Figure 2. The

cdf graph tend to move at a constant value from 0 to 1 on the x-axis before projecting to one on the y-axis

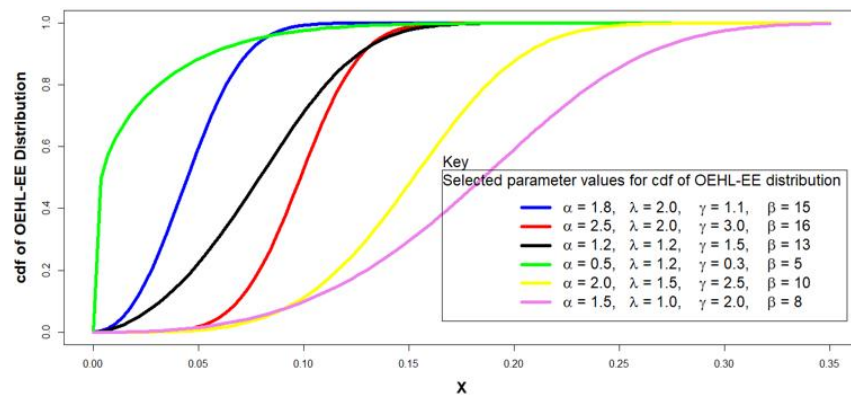


Figure 2: Plot of cdf for some selected parameter values of OEHL-EE distribution

Plots of hazard function (10) for the OEHL-EE distribution are displayed in Figure 3. It was examine from Figure 3 show that the shape of the $h(x)$ of the OEHL-EE shows the hazard functions can be constant,

increasing or decreases, depending on the shape and scale parameters.

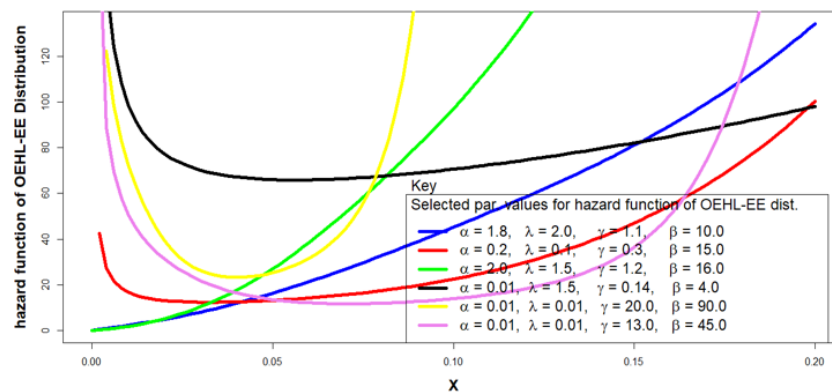


Figure 3: Plot of hazard function for some selected parameter values of OEHL-EE distribution

Plots of survival function (13) for the OEHL-EE distribution are displayed in Figure 4. It was deduced from Figure 4 that the shape parameters has strong effect on the survival plot. A decrease in the shape

parameter tend to reduce the movement of the line on the y-axis and no plot is exceeding the bench mark of 1:

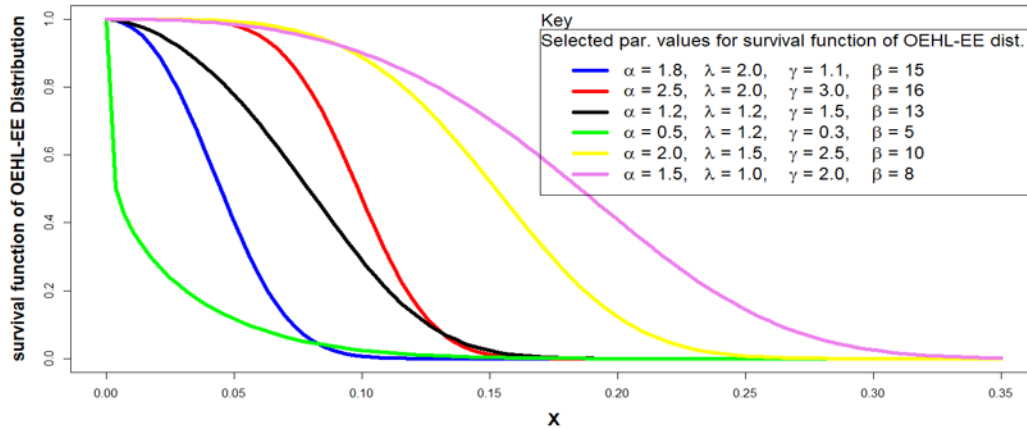


Figure 4: Plot of survival function for some selected parameter values of OEHL-EE distribution

III. MATHEMATICAL PROPERTIES

This section provide some mathematical properties of the OEHL-EE

Moment about origin for the proposed OEHL-EE Distribution

The k^{th} moments of OEHL-EE distribution is given as

$$\mu'_k = E(X^k) = \frac{2\alpha\Gamma(\frac{k}{\gamma} + \alpha)}{\beta^k(2^{(\alpha+1)})\lambda^{(\frac{k}{\gamma})}} \quad (13)$$

The k^{th} Moments for $k = 1, 2, 3, \text{ and } 4$

Using equations (13), the corresponding moments for $k = 1, 2, 3, \text{ and } 4$ are as follows

for $k = 1$

$$E(X) = \frac{2\alpha\Gamma(\frac{1}{\gamma} + \alpha)}{\beta(2^{(\alpha+1)})\lambda^{(\frac{1}{\gamma})}} \quad (14)$$

for $k = 2$

$$E(X^2) = \frac{2\alpha\Gamma(\frac{2}{\gamma} + \alpha)}{\beta^2(2^{(\alpha+1)})\lambda^{(\frac{2}{\gamma})}} \quad (15)$$

for $k = 3$

$$E(X^3) = \frac{2\alpha\Gamma(\frac{3}{\gamma} + \alpha)}{\beta^3(2^{(\alpha+1)})\lambda^{(\frac{3}{\gamma})}} \quad (16)$$

for $k = 4$

$$E(X^4) = \frac{2\alpha\Gamma(\frac{4}{\gamma} + \alpha)}{\beta^4(2^{(\alpha+1)})\lambda^{(\frac{4}{\gamma})}} \quad (17)$$

Mean and Variance of OEHL-EE Distribution

Mean of OEHL-EE

The mean of a given random variable X is given as

$$E(X) = \mu = \mu'_1 \quad (18)$$

Substituting equation (14) in equation (18), we have the mean of OEHL-EE distribution; that is, the first moment about origin for OEHL-EE distribution is the mean of OEHL-EE. It is defined as

$$\mu'_1 = E(X) = \frac{2\alpha\Gamma(\frac{1}{\gamma} + \alpha)}{\beta(2^{(\alpha+1)})\lambda^{(\frac{1}{\gamma})}} \quad (19)$$

Variance of OEHL-EE

The variance of a given random variable X is given as

$$\text{Var}(X) = \mu'_2 - (\mu'_1)^2 \quad (20)$$

Substituting equations (14) and (15) in equation (20), we have the variance of OEHL-EE distribution. That

is, the second moment of OEHL-EE minus the first moment of it; and it is defined as

$$\text{Var}(X) = \frac{2\alpha\Gamma\left(\frac{2}{\gamma} + \alpha\right)}{\beta^2(2(\alpha+1))\lambda^{\left(\frac{2}{\gamma}\right)}} - \left\{ \frac{2\alpha\Gamma\left(\frac{1}{\gamma} + \alpha\right)}{\beta(2(\alpha+1))\lambda^{\left(\frac{1}{\gamma}\right)}} \right\}^2 \quad (21)$$

The j^{th} Moment about the Mean for OEHL-EE Distribution

The j^{th} moment about the mean for OEHL-EE distribution is given as

$$E(x - \mu)^j = \sum_{k=0}^j \binom{j}{k} (-u)^{j-k} \left(\frac{2\alpha\Gamma\left(\frac{k}{\gamma} + \alpha\right)}{\beta^k(2(\alpha+1))\lambda^{\left(\frac{k}{\gamma}\right)}} \right) \quad (22)$$

The j^{th} Moment about the Mean for $j = 1, 2, 3$, and 4
for $j = 1$

$$E(x - \mu) = \sum_{k=0}^1 \binom{1}{k} (-u)^{1-k} \left(\frac{2\alpha\Gamma\left(\frac{k}{\gamma} + \alpha\right)}{\beta^k(2(\alpha+1))\lambda^{\left(\frac{k}{\gamma}\right)}} \right) \quad (23)$$

for $j = 2$

$$E(x - \mu)^2 = \sum_{k=0}^2 \binom{2}{k} (-u)^{2-k} \left(\frac{2\alpha\Gamma\left(\frac{k}{\gamma} + \alpha\right)}{\beta^k(2(\alpha+1))\lambda^{\left(\frac{k}{\gamma}\right)}} \right) \quad (24)$$

for $j = 3$

$$E(x - \mu)^3 = \sum_{k=0}^3 \binom{3}{k} (-u)^{3-k} \left(\frac{2\alpha\Gamma\left(\frac{k}{\gamma} + \alpha\right)}{\beta^k(2(\alpha+1))\lambda^{\left(\frac{k}{\gamma}\right)}} \right) \quad (25)$$

$$\int_0^\infty f(x; \alpha, \lambda, \gamma, \beta) dx$$

$$= \int_0^\infty \frac{2\alpha\lambda \left(\gamma\beta(\exp(-\beta x))(1 - \exp(-\beta x))^{\gamma-1} \right) \left[\exp\left(\frac{-\lambda(1 - \exp(-\beta x))^\gamma}{1 - (1 - \exp(-\beta x))^\gamma}\right) \right] \left[1 - \exp\left(\frac{-\lambda(1 - \exp(-\beta x))^\gamma}{1 - (1 - \exp(-\beta x))^\gamma}\right) \right]^{\alpha-1}}{\left[1 - (1 - \exp(-\beta x))^\gamma \right]^2 \left[1 + \exp\left(\frac{-\lambda(1 - \exp(-\beta x))^\gamma}{1 - (1 - \exp(-\beta x))^\gamma}\right) \right]^{\alpha+1}} dx$$

$x \in (0, \infty); \alpha, \lambda, \gamma, \text{ and } \beta > 0$

The moment generating function (MGF) of the OEHL-EE Distribution is given as

for $j = 4$

$$E(x - \mu)^4 = \sum_{k=0}^4 \binom{4}{k} (-u)^{4-k} \left(\frac{2\alpha\Gamma\left(\frac{k}{\gamma} + \alpha\right)}{\beta^k(2(\alpha+1))\lambda^{\left(\frac{k}{\gamma}\right)}} \right) \quad (26)$$

Coefficient of Variation (CV)

The coefficient of variation (CV) is defined as

$$CV = \frac{\sqrt{\sigma^2}}{\mu} \quad (27)$$

Recall that the second moment about the mean is equal to variance, σ^2 and the first moment about the origin is equal to mean, μ ; then using equations (8) and (18) the coefficient of variation of OEHL-EE distribution is given as

$$CV = \frac{\left(\beta(2(\alpha+1))\lambda^{\left(\frac{1}{\gamma}\right)} \right) \sqrt{\sum_{k=0}^2 \binom{2}{k} (-u)^{2-k} \left(\frac{2\alpha\Gamma\left(\frac{k}{\gamma} + \alpha\right)}{\beta^k(2(\alpha+1))\lambda^{\left(\frac{k}{\gamma}\right)}} \right)}}{2\alpha\Gamma\left(\frac{1}{\gamma} + \alpha\right)} \quad (28)$$

Moment Generating Function (MGF) of OEHL-EE Distribution

Moment Generating Function (MGF) is a way of organizing all the moments into one mathematical quantity. The moment-generating function of a random variable X is given as

$$M_x(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x) dx \quad (29)$$

Let X be a random variable with the OEHL-EE distribution with parameters α, λ, γ and β

$$M_x(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \frac{2\alpha\Gamma\left(\frac{k}{\gamma} + \alpha\right)}{\beta^k(2(\alpha+1))\lambda^{\left(\frac{k}{\gamma}\right)}} \quad (30)$$

Rényi Entropy of the OEHL-EE Distribution

Assume X is a random variable that follows OEHL-EE distribution with parameters α, λ, γ and β ; that is, $X \sim \text{OEHL-EE}(\alpha, \lambda, \gamma, \beta)$.

$$I_{\theta}(X) = \frac{1}{1-\theta} \log \left\{ \frac{(2\alpha\lambda\gamma\beta)^{\theta}}{\beta} \sum_{j=0}^{\infty} \frac{\theta^j}{j!} (-\lambda)^k \binom{\theta(\alpha-1)}{n} \binom{-\theta(\alpha+1)}{m} \binom{k+2\theta+l-1}{l} \right\} \frac{\Gamma(\theta)\Gamma(\theta(\gamma-1)+\gamma j+1)}{\Gamma(\theta+\theta(\gamma-1)+\gamma j+1)} \quad (31)$$

Order Statistics

Order statistics refer to a sample of random variables arranged in ascending order, from the smallest to the largest value. It plays an important role in various statistical applications, including robust statistical estimation, outlier detection, probability distribution characterization, goodness-of-fit tests, entropy estimation, censored samples analysis, reliability assessment, quality control, and material strength evaluation. Consider a random sample $X_1, X_2, \dots, X_n$ drawn from a distribution with probability density function (pdf), $f(x)$. The corresponding order statistics, denoted as $X_{1:n}, X_{2:n}, \dots, X_{i:n}$, represent the

The Rényi Entropy of a random variable X that follows OEHL-EE distribution is given by

sorted values of the sample, where $X_{(1:n)}$ is the smallest observation and $X_{(n:n)}$ is the largest.

The pdf, $f_{i:n}(x)$ of the i^{th} order statistic can be defined as

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} f(x) F(x)^{i-1} [1-F(x)]^{n-i} \quad (32)$$

Order Statistic of OEHL-EE Distribution

The density function of the k^{th} order statistics for a given random samples $x_1, x_2, \dots, x_n$ from the cdf and pdf of OEHL-EE distribution is given as

$$f_{k:n}(x) = \frac{n!}{(k-1)!(n-k)!} \frac{2\alpha\lambda(\gamma\beta(e^{-\beta x})(1-e^{-\beta x})^{\gamma-1}) \left[\exp\left(\frac{-\lambda(1-e^{-\beta x})^{\gamma}}{1-(1-e^{-\beta x})^{\gamma}}\right) \right] \left[1 - \exp\left(\frac{-\lambda(1-e^{-\beta x})^{\gamma}}{1-(1-e^{-\beta x})^{\gamma}}\right) \right]^{\alpha-1}}{\left[1 - (1-e^{-\beta x})^{\gamma} \right]^2 \left[1 + \exp\left(\frac{-\lambda(1-e^{-\beta x})^{\gamma}}{1-(1-e^{-\beta x})^{\gamma}}\right) \right]^{\alpha+1}} \times \left[\left(\frac{1 - \exp\left(\frac{-\lambda(1-e^{-\beta x})^{\gamma}}{1-(1-e^{-\beta x})^{\gamma}}\right)}{1 + \exp\left(\frac{-\lambda(1-e^{-\beta x})^{\gamma}}{1-(1-e^{-\beta x})^{\gamma}}\right)} \right)^{\alpha} \right]^{k-1} \left[1 - \left(\frac{1 - \exp\left(\frac{-\lambda(1-e^{-\beta x})^{\gamma}}{1-(1-e^{-\beta x})^{\gamma}}\right)}{1 + \exp\left(\frac{-\lambda(1-e^{-\beta x})^{\gamma}}{1-(1-e^{-\beta x})^{\gamma}}\right)} \right)^{\alpha} \right]^{n-k} \quad (33)$$

Two approaches are used in obtaining the Skewness and kurtosis of the OEHL-EE distribution. These approaches include the measure of kurtosis (k.u) and skewness (s.k) based on moments and quantiles. In the moments based approach,

$$CS = \frac{\sum_{k=0}^3 \binom{3}{k} (-u)^{3-k} \left(\frac{2\alpha\Gamma(\frac{k}{\gamma}+\alpha)}{\beta^k (2(\alpha+1)) \lambda^{\frac{k}{\gamma}}} \right)}{\left(\sum_{k=0}^2 \binom{2}{k} (-u)^{2-k} \left(\frac{2\alpha\Gamma(\frac{k}{\gamma}+\alpha)}{\beta^k (2(\alpha+1)) \lambda^{\frac{k}{\gamma}}} \right) \right)^{\frac{3}{2}}} \quad (34)$$

$$CK = \frac{\sum_{k=0}^3 \binom{4}{k} (-u)^{4-k} \left(\frac{2\alpha\Gamma(\frac{k}{\gamma}+\alpha)}{\beta^k (2(\alpha+1)) \lambda^{\frac{k}{\gamma}}} \right)}{\left(\sum_{k=0}^2 \binom{2}{k} (-u)^{2-k} \left(\frac{2\alpha\Gamma(\frac{k}{\gamma}+\alpha)}{\beta^k (2(\alpha+1)) \lambda^{\frac{k}{\gamma}}} \right) \right)^2} \quad (35)$$

IV. PARAMETER ESTIMATION

From parameter estimation obtain the maximum likelihood estimator for OEHL-EE in the case of complete sample and simulation study is performed. Parameters Estimation of the OEHL-EE Distribution using Maximum Likelihood Estimation (MLE) Method

Let $x_1, x_2, \dots, x_n$ be the random sample of size n from the OEHL-EE distribution with unknown parameters α, λ, γ and β .

Let $\theta = (\alpha, \lambda, \gamma, \beta)^T$ be the $p \times 1$ parameter vector. The pdf of OEHL-EE distribution is given as

$$f(x; \alpha, \lambda, \gamma, \beta) = \frac{2\alpha\lambda(\gamma\beta(\exp(-\beta x))(1-\exp(-\beta x))^{\gamma-1}) \left[\exp\left(\frac{-\lambda(1-\exp(-\beta x))^{\gamma}}{1-(1-\exp(-\beta x))^{\gamma}}\right) \right] \left[1-\exp\left(\frac{-\lambda(1-\exp(-\beta x))^{\gamma}}{1-(1-\exp(-\beta x))^{\gamma}}\right) \right]^{\alpha-1}}{\left[1-(1-\exp(-\beta x))^{\gamma} \right]^2 \left[1+\exp\left(\frac{-\lambda(1-\exp(-\beta x))^{\gamma}}{1-(1-\exp(-\beta x))^{\gamma}}\right) \right]^{\alpha+1}} \quad (36)$$

$$x \in (0, \infty); \alpha, \lambda, \gamma, \beta > 0$$

For determining the MLE of θ using (36)

where, α, λ , and γ are the shape parameters; β is the scale parameter.

$$L(\theta) = \prod_{i=1}^n f(x_i; \alpha, \lambda, \gamma, \beta)$$

$$= \prod_{i=1}^n \left(\frac{2\alpha\lambda(\gamma\beta e^{-\beta x_i}(1-e^{-\beta x_i})^{\gamma-1}) \left[\exp\left(\frac{-\lambda(1-e^{-\beta x_i})^{\gamma}}{1-(1-e^{-\beta x_i})^{\gamma}}\right) \right] \left[1-\exp\left(\frac{-\lambda(1-e^{-\beta x_i})^{\gamma}}{1-(1-e^{-\beta x_i})^{\gamma}}\right) \right]^{\alpha-1}}{\left[1-(1-e^{-\beta x_i})^{\gamma} \right]^2 \left[1+\exp\left(\frac{-\lambda(1-e^{-\beta x_i})^{\gamma}}{1-(1-e^{-\beta x_i})^{\gamma}}\right) \right]^{\alpha+1}} \right)$$

$$= \frac{2^n \alpha^n \lambda^n \gamma^n \beta^n \prod_{i=1}^n e^{-\beta x_i} \prod_{i=1}^n (1-e^{-\beta x_i})^{\gamma-1} \prod_{i=1}^n \left[\exp\left(\frac{-\lambda(1-e^{-\beta x_i})^{\gamma}}{1-(1-e^{-\beta x_i})^{\gamma}}\right) \right] \prod_{i=1}^n \left[1-\exp\left(\frac{-\lambda(1-e^{-\beta x_i})^{\gamma}}{1-(1-e^{-\beta x_i})^{\gamma}}\right) \right]^{\alpha-1}}{\prod_{i=1}^n \left[1-(1-e^{-\beta x_i})^{\gamma} \right]^2 \prod_{i=1}^n \left[1+\exp\left(\frac{-\lambda(1-e^{-\beta x_i})^{\gamma}}{1-(1-e^{-\beta x_i})^{\gamma}}\right) \right]^{\alpha+1}} \quad (37)$$

Taking the log-likelihood of (3.163), i.e., $\log L(\theta)$, we have

$$\log L(\theta) = \log \left(\frac{2^n \alpha^n \lambda^n \gamma^n \beta^n \prod_{i=1}^n e^{-\beta x_i} \prod_{i=1}^n (1-e^{-\beta x_i})^{\gamma-1} \prod_{i=1}^n \left[\exp\left(\frac{-\lambda(1-e^{-\beta x_i})^{\gamma}}{1-(1-e^{-\beta x_i})^{\gamma}}\right) \right] \prod_{i=1}^n \left[1-\exp\left(\frac{-\lambda(1-e^{-\beta x_i})^{\gamma}}{1-(1-e^{-\beta x_i})^{\gamma}}\right) \right]^{\alpha-1}}{\prod_{i=1}^n \left[1-(1-e^{-\beta x_i})^{\gamma} \right]^2 \prod_{i=1}^n \left[1+\exp\left(\frac{-\lambda(1-e^{-\beta x_i})^{\gamma}}{1-(1-e^{-\beta x_i})^{\gamma}}\right) \right]^{\alpha+1}} \right) \quad (38)$$

$$= n \log 2 + n \log \alpha + n \log \lambda + n \log \gamma + n \log \beta - \beta \sum_{i=1}^n x_i + (\gamma - 1) \sum_{i=1}^n \log(1 - e^{-\beta x_i}) + \sum_{i=1}^n \left(\frac{-\lambda(1-e^{-\beta x_i})^{\gamma}}{1-(1-e^{-\beta x_i})^{\gamma}} \right) + (\alpha - 1) \sum_{i=1}^n \log \left(1 - \exp \left(\frac{-\lambda(1-e^{-\beta x_i})^{\gamma}}{1-(1-e^{-\beta x_i})^{\gamma}} \right) \right) - 2 \sum_{i=1}^n \log(1 - (1 - e^{-\beta x_i})^{\gamma}) - (\alpha + 1) \sum_{i=1}^n \log \left(1 + \exp \left(\frac{-\lambda(1-e^{-\beta x_i})^{\gamma}}{1-(1-e^{-\beta x_i})^{\gamma}} \right) \right) \quad (39)$$

Differentiating (39) with respect to each parameter α, λ, γ , and β , and equating each of the derivative to zero

$$\frac{d \log L(\theta)}{d \alpha} = \frac{n}{\alpha} + \sum_{i=1}^n \log \left(1 - \exp \left(\frac{-\lambda(1-e^{-\beta x_i})^{\gamma}}{1-(1-e^{-\beta x_i})^{\gamma}} \right) \right) - \sum_{i=1}^n \log \left(1 + \exp \left(\frac{-\lambda(1-e^{-\beta x_i})^{\gamma}}{1-(1-e^{-\beta x_i})^{\gamma}} \right) \right) = 0 \quad (40)$$

$$\frac{d \log L(\theta)}{d \lambda} = \frac{n}{\lambda} - \sum_{i=1}^n \frac{(1-e^{-\beta x_i})^{\gamma}}{(1-(1-e^{-\beta x_i})^{\gamma})^2} \left\{ \exp \left(\frac{-\lambda(1-e^{-\beta x_i})^{\gamma}}{1-(1-e^{-\beta x_i})^{\gamma}} \right) - \frac{(\alpha-1) \exp \left(\frac{-\lambda(1-e^{-\beta x_i})^{\gamma}}{1-(1-e^{-\beta x_i})^{\gamma}} \right)}{1-\exp \left(\frac{-\lambda(1-e^{-\beta x_i})^{\gamma}}{1-(1-e^{-\beta x_i})^{\gamma}} \right)} - \frac{(\alpha+1)}{1+\exp \left(\frac{-\lambda(1-e^{-\beta x_i})^{\gamma}}{1-(1-e^{-\beta x_i})^{\gamma}} \right)} \right\} = 0 \quad (41)$$

$$\begin{aligned} \frac{d \log L(\theta)}{d\gamma} &= \frac{n}{\gamma} + \sum_{i=1}^n \log(1 - e^{-\beta x_i}) - \\ &2 \sum_{i=1}^n \frac{(1 - e^{-\beta x_i})^\gamma \log(1 - e^{-\beta x_i})}{1 - (1 - e^{-\beta x_i})^\gamma} - \\ &\sum_{i=1}^n \frac{\lambda(1 - e^{-\beta x_i})^\gamma \log(1 - e^{-\beta x_i})}{1 - (1 - e^{-\beta x_i})^{\gamma^2}} \left\{ \exp\left(\frac{-\lambda(1 - e^{-\beta x_i})^\gamma}{1 - (1 - e^{-\beta x_i})^\gamma}\right) - \right. \\ &\left. \frac{(\alpha - 1) \exp\left(\frac{-\lambda(1 - e^{-\beta x_i})^\gamma}{1 - (1 - e^{-\beta x_i})^\gamma}\right)}{1 - \exp\left(\frac{-\lambda(1 - e^{-\beta x_i})^\gamma}{1 - (1 - e^{-\beta x_i})^\gamma}\right)} - \frac{(\alpha + 1)}{1 + \exp\left(\frac{-\lambda(1 - e^{-\beta x_i})^\gamma}{1 - (1 - e^{-\beta x_i})^\gamma}\right)} \right\} = 0 \end{aligned} \quad (42)$$

$$\begin{aligned} \frac{d \log L(\theta)}{d\beta} &= \frac{n}{\beta} - \sum_{i=1}^n x_i + (\gamma - 1) \sum_{i=1}^n \frac{e^{-\beta x_i x_i}}{1 - e^{-\beta x_i}} - \\ &2 \sum_{i=1}^n \frac{\gamma e^{-\beta x_i x_i} (1 - e^{-\beta x_i})^{\gamma-1}}{1 - (1 - e^{-\beta x_i})^\gamma} = 0 \end{aligned} \quad (43)$$

By setting the nonlinear system of equations to zero, that is, $\frac{d \log L(\theta)}{d\alpha} = \frac{d \log L(\theta)}{d\lambda} = \frac{d \log L(\theta)}{d\gamma} = \frac{d \log L(\theta)}{d\beta} = 0$ and solving them simultaneously gives the MLE $\theta = (\alpha, \lambda, \gamma, \beta)^T$. Moreover, numerical computations are needed to find the solution of this nonlinear system of equations, $\frac{d \log L(\theta)}{d\alpha} = \frac{d \log L(\theta)}{d\lambda} = \frac{d \log L(\theta)}{d\gamma} = \frac{d \log L(\theta)}{d\beta} = 0$ and it is usually more convenient to obtain by using nonlinear optimization methods such as the Newton-Rapson algorithm.

Asymptotic Confidence Intervals

In the asymptotic scenario, the distribution of the maximum likelihood estimators (MLE) $\hat{\alpha}, \hat{\lambda}, \hat{\gamma}, \hat{\beta}$ tends to be approximately multivariate normal. Therefore, as the sample size, n , approaches infinity (as $n \rightarrow \infty$), the asymptotic distribution of the MLE $\hat{\alpha}, \hat{\lambda}, \hat{\gamma}, \hat{\beta}$ is described by

$$\begin{pmatrix} \hat{\alpha} \\ \hat{\lambda} \\ \hat{\gamma} \\ \hat{\beta} \end{pmatrix} \sim N \left[\begin{pmatrix} \hat{\alpha} \\ \hat{\lambda} \\ \hat{\gamma} \\ \hat{\beta} \end{pmatrix}, \begin{pmatrix} \hat{V}_{11} & \hat{V}_{12} & \hat{V}_{13} & \hat{V}_{14} \\ \hat{V}_{21} & \hat{V}_{22} & \hat{V}_{23} & \hat{V}_{24} \\ \hat{V}_{31} & \hat{V}_{32} & \hat{V}_{33} & \hat{V}_{34} \\ \hat{V}_{41} & \hat{V}_{42} & \hat{V}_{43} & \hat{V}_{44} \end{pmatrix} \right]$$

Where

$$V_{ij}^{-1} = -E \begin{bmatrix} \frac{\partial^2 L(\theta)}{\partial \alpha^2} & \frac{\partial^2 L(\theta)}{\partial \alpha \partial \lambda} & \frac{\partial^2 L(\theta)}{\partial \alpha \partial \gamma} & \frac{\partial^2 L(\theta)}{\partial \alpha \partial \beta} \\ \frac{\partial^2 L(\theta)}{\partial \lambda \partial \alpha} & \frac{\partial^2 L(\theta)}{\partial \lambda^2} & \frac{\partial^2 L(\theta)}{\partial \lambda \partial \gamma} & \frac{\partial^2 L(\theta)}{\partial \lambda \partial \beta} \\ \frac{\partial^2 L(\theta)}{\partial \gamma \partial \alpha} & \frac{\partial^2 L(\theta)}{\partial \gamma \partial \lambda} & \frac{\partial^2 L(\theta)}{\partial \gamma^2} & \frac{\partial^2 L(\theta)}{\partial \gamma \partial \beta} \\ \frac{\partial^2 L(\theta)}{\partial \beta \partial \alpha} & \frac{\partial^2 L(\theta)}{\partial \beta \partial \lambda} & \frac{\partial^2 L(\theta)}{\partial \beta \partial \gamma} & \frac{\partial^2 L(\theta)}{\partial \beta^2} \end{bmatrix}; \quad i, j = 1, 2, 3, 4 \quad (44)$$

Solving the inverse matrix of the observed information matrix in equation (44) gives the asymptotic variance and co-variance of the maximum likelihood estimators $\hat{\alpha}, \hat{\lambda}, \hat{\gamma}$, and $\hat{\beta}$. The approximate $(1 - \alpha)100\%$ asymptotic confidence intervals (C. I) for $\hat{\alpha}, \hat{\lambda}, \hat{\gamma}$, and $\hat{\beta}$ are given by

$$\begin{aligned} \hat{\alpha} \pm Z_{\frac{\alpha}{2}} \sqrt{\hat{V}_{11}}; \quad \hat{\lambda} \pm Z_{\frac{\alpha}{2}} \sqrt{\hat{V}_{22}}; \quad \hat{\gamma} \pm Z_{\frac{\alpha}{2}} \sqrt{\hat{V}_{33}}; \quad \hat{\beta} \\ \pm Z_{\frac{\alpha}{2}} \sqrt{\hat{V}_{44}} \end{aligned}$$

where $Z_{\frac{\alpha}{2}}$ is the upper α th percentile of the standard normal distribution.

V. MONTE CARLO SIMULATION STUDY

5.1 Simulation Design

A Monte Carlo simulation study was conducted to examine the performance of the maximum likelihood estimators of the four parameters of the OEHL-EE distribution.

The simulation considered sample sizes $n=20, 50, 80, 120, 200, 300, 500$. For every sample size, the simulation was replicated 1,000 times. The simulated observations were generated using the quantile function of the OEHL-EE distribution in equation 45. The estimators were obtained using numerical optimization through the optim function in R version 4.5.1. Their performance was assessed using the maximum likelihood estimate, bias, variance and mean squared error.

$$x_U = -\frac{1}{\beta} \ln \left\{ 1 - \left(\frac{-\ln(1 - u^{\frac{1}{\alpha}}) + \ln(1 + u^{\frac{1}{\alpha}})}{\lambda - \ln(1 - u^{\frac{1}{\alpha}}) + \ln(1 + u^{\frac{1}{\alpha}})} \right)^{\frac{1}{\gamma}} \right\} \quad (45)$$

where, u has uniform distribution over the interval $(0, 1)$; i.e., $u \sim U(0, 1)$.

5.2 Simulation Results

The simulation results reported show the ML estimates, biases, variances and MSEs for the four parameters across the selected sample sizes.

Table 1: The ML estimates, biases, variances and MSEs of the OEHL-EE distribution for some selected values of $\alpha = 2.5$, $\lambda = 1.2$, $\gamma = 1.8$, and $\beta = 0.5$

Sample size (n)	Parameter	ML Estimate	Bias	Variance	MSE
20	$\hat{\alpha}$	0.3988	2.1012	0.0427	4.4577
	$\hat{\lambda}$	0.1223	1.0777	0.1070	1.2684
	$\hat{\gamma}$	0.9137	0.8863	4.2320	5.0175
	$\hat{\beta}$	1.7537	1.2537	0.8533	2.4251
50	$\hat{\alpha}$	1.7428	0.7572	2.3368	2.9102
	$\hat{\lambda}$	0.1202	1.0798	0.1009	1.2669
	$\hat{\gamma}$	0.9222	0.8778	4.2300	5.0005
	$\hat{\beta}$	0.8953	0.3953	0.1448	0.3011
80	$\hat{\alpha}$	1.7451	0.7549	2.2472	2.8171
	$\hat{\lambda}$	0.7644	0.4356	1.1096	1.2993
	$\hat{\gamma}$	0.9407	0.8593	2.8744	3.6128
	$\hat{\beta}$	0.3210	0.1790	0.0173	0.0493
120	$\hat{\alpha}$	2.0425	0.4575	1.4021	1.6114
	$\hat{\lambda}$	0.2162	0.9838	0.2674	1.2353
	$\hat{\gamma}$	1.3301	0.4699	2.3120	2.5328
	$\hat{\beta}$	0.5241	0.0241	0.0162	0.0168
200	$\hat{\alpha}$	2.3392	0.1608	0.3326	0.3585
	$\hat{\lambda}$	1.7921	0.5921	0.4001	0.7507
	$\hat{\gamma}$	1.6005	0.1995	1.2211	1.2609
	$\hat{\beta}$	0.3254	0.1746	0.1716	0.2021
300	$\hat{\alpha}$	3.0903	0.5903	1.1011	1.4496
	$\hat{\lambda}$	1.6274	0.4274	1.1083	1.2910
	$\hat{\gamma}$	1.7612	0.0388	0.1418	0.1433
	$\hat{\beta}$	0.6659	0.1659	0.0212	0.0487
500	$\hat{\alpha}$	3.2338	0.7338	1.7181	2.2566
	$\hat{\lambda}$	1.0071	0.1929	1.3878	1.4250
	$\hat{\gamma}$	2.0172	0.2172	1.2435	1.2907
	$\hat{\beta}$	0.5677	0.0677	0.0058	0.0104

Source: Simulated data of OEHL-EE

Table 2: The ML estimates, biases, variances and MSEs of the OEHL-EE distribution for the some selected values of $\alpha = 7$, $\lambda = 4$, $\gamma = 5$, and $\beta = 0.9$

Sample size (n)	Parameter	ML Estimate	Bias	Variance	MSE
20	$\hat{\alpha}$	3.3220	3.6780	6.9207	20.4484
	$\hat{\lambda}$	1.4993	2.5007	5.2866	11.5401

	$\hat{\gamma}$	3.6097	1.3903	3.7374	5.6703
	$\hat{\beta}$	1.8253	0.9253	3.6288	4.4850
50	$\hat{\alpha}$	3.3700	3.6300	5.6595	18.8364
	$\hat{\lambda}$	2.1562	1.8438	3.0401	6.4397
	$\hat{\gamma}$	4.5308	0.4692	1.3621	1.5822
	$\hat{\beta}$	1.1280	0.2280	0.4837	0.5357
80	$\hat{\alpha}$	3.6188	3.3812	3.9382	15.3707
	$\hat{\lambda}$	2.4173	1.5827	2.0175	4.5224
	$\hat{\gamma}$	4.8365	0.1635	0.8189	0.8456
	$\hat{\beta}$	0.4803	0.4197	0.2604	0.4365
120	$\hat{\alpha}$	4.1762	2.8238	3.0366	11.0104
	$\hat{\lambda}$	3.0277	0.9723	1.9348	2.8802
	$\hat{\gamma}$	5.5038	0.5038	1.1928	1.4466
	$\hat{\beta}$	1.0063	0.1063	0.2164	0.2277
200	$\hat{\alpha}$	4.2456	2.7544	3.2941	10.8808
	$\hat{\lambda}$	4.6849	0.6849	1.4850	1.9541
	$\hat{\gamma}$	5.5347	0.5347	1.5726	1.8585
	$\hat{\beta}$	0.3851	0.5149	0.1391	0.4042
300	$\hat{\alpha}$	4.8012	2.1988	2.0642	6.8989
	$\hat{\lambda}$	5.3930	1.3930	2.6608	4.6012
	$\hat{\gamma}$	6.3770	1.3770	2.1653	4.0614
	$\hat{\beta}$	1.0024	0.1024	0.0833	0.0938
500	$\hat{\alpha}$	6.4979	0.5021	1.5627	1.8148
	$\hat{\lambda}$	5.5627	1.5627	2.9940	5.4360
	$\hat{\gamma}$	8.0663	3.0663	3.0118	12.4140
	$\hat{\beta}$	0.9188	0.0188	0.1099	0.1103

Source: Simulated data of OEHL-EE

5.3 Interpretation of the Simulation Study

The simulation results demonstrate that the ML estimation procedure can be implemented for the OEHL-EE model across a range of sample sizes. The thesis uses the simulation to examine how the estimates, bias and MSE behave as the sample size increases.

The broader simulation investigation also examines how the parameters affect the median, mean, variance, coefficient of variation, skewness and kurtosis of the proposed distribution. The study reports that the parameters have different effects on the location, dispersion and shape characteristics of the model.

These results demonstrate the flexibility of the OEHL-EE distribution and provide numerical evidence

supporting the use of maximum likelihood estimation for the proposed model.

VI. DISCUSSION

The OEHL-EE distribution provides a flexible four-parameter extension of the Exponentiated Exponential distribution through the Odd Exponentiated Half-Logistic-G generator framework.

One important feature of the proposed model is the availability of a closed-form quantile function, which facilitates random-number generation and simulation. The quantile representation was directly used in the simulation implementation contained in the thesis.

The theoretical investigation also demonstrates that the model possesses a broad range of statistical

characteristics. In particular, the derivation of moments, skewness, kurtosis, entropy and order statistics provides a comprehensive framework for studying the distribution.

The hazard function is particularly relevant for lifetime and reliability applications. The thesis illustrates a bathtub-shaped hazard function for selected parameter configurations, indicating that the model can represent a failure-rate pattern consisting of early failures, a relatively stable useful-life period and increasing failures associated with wear-out.

The simulation investigation further demonstrates the practical feasibility of estimating the four unknown parameters using maximum likelihood estimation. The use of numerical optimization is appropriate because the resulting likelihood equations are nonlinear.

VII. CONCLUSION

A new four-parameter lifetime distribution, termed the Odd Exponentiated Half-Logistic Exponentiated Exponential distribution, has been presented. The model extends the Exponentiated Exponential distribution through the Odd Exponentiated Half-Logistic-G generator.

Several important properties of the proposed distribution were established, including the probability density function, cumulative distribution function, survival function, hazard function, quantile function, moments, mean, variance, coefficient of variation, skewness, kurtosis, moment generating function, Rényi entropy and order statistics.

Maximum likelihood estimation was employed for estimating the model parameters, while asymptotic confidence intervals were formulated using the observed information matrix. Because the likelihood equations are nonlinear, numerical optimization was employed.

A Monte Carlo simulation study involving sample sizes of 20, 50, 80, 120, 200, 300 and 500, with 1,000 replications for each sample size, was used to examine the performance of the estimators. The simulation results provide numerical evidence concerning the

behaviour of the maximum likelihood estimates, bias, variance and mean squared error.

The theoretical and computational results indicate that the OEHL-EE distribution provides a flexible framework for modelling lifetime and reliability data. Further work may investigate additional estimation procedures, Bayesian inference, regression extensions and applications to censored survival data.

Declarations

Data Availability

The theoretical and simulation results reported in this manuscript are based on the research materials developed in the underlying doctoral study. Simulation procedures and parameter configurations are described in the manuscript.

Conflict of Interest

The author declares that there is no conflict of interest.

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