

A Flexible Odd Exponentiated Half-Logistic Exponentiated Exponential Model for Lifetime and Reliability Data: Applications to Real-Life Datasets

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Abstract- Flexible probability distributions are important for modelling real-life data exhibiting skewness, kurtosis and diverse distributional characteristics. This study investigates the applicability of the Odd Exponentiated Half-Logistic Exponentiated Exponential (OEHL-EE) distribution, a four-parameter extension of the Exponentiated Exponential distribution constructed using the Odd Exponentiated Half-Logistic-G generator. Although the theoretical properties of the OEHL-EE distribution have been established, its practical performance is examined in this study through applications to four real-life datasets. The datasets comprise monthly inflation rates in Nigeria from January 2003 to June 2023, fatigue-fracture lifetimes of Kevlar 373/epoxy specimens, the Gross and Clark dataset, and turbocharger failure times. The proposed model is compared with several competing distributions using maximum likelihood estimation and a collection of model-selection and goodness-of-fit criteria, including the negative log-likelihood, Akaike Information Criterion, Consistent Akaike Information Criterion, Hannan–Quinn Information Criterion, Bayesian Information Criterion, Anderson–Darling, Cramér–von Mises and Kolmogorov–Smirnov statistics. The results reported in the study show that the OEHL-EE model provides a close fit to the four datasets and generally records favourable goodness-of-fit measures relative to the competing models considered. Graphical comparisons based on fitted density and cumulative distribution functions further demonstrate the ability of the proposed model to describe the empirical behaviour of the datasets. The findings illustrate the practical usefulness of the OEHL-EE distribution for modelling skewed lifetime, reliability and other positive-valued data.

Keywords: Odd Exponentiated Half-Logistic; Exponentiated Exponential; lifetime data; reliability; maximum likelihood estimation; goodness of fit; model selection.

I. INTRODUCTION

Probability distributions provide essential tools for describing uncertainty and modelling observed phenomena in statistics, reliability engineering, economics and other quantitative disciplines. In practical applications, however, commonly used distributions may not adequately capture the skewness, kurtosis or other features present in observed datasets. This has motivated the development of generalized and extended probability models with additional shape parameters capable of providing greater modelling flexibility.

The Exponentiated Exponential (EE) distribution is an extension of the classical exponential distribution that provides additional flexibility through an extra shape parameter. Further flexibility can be obtained by embedding the baseline distribution within generator families. One such construction is the Odd Exponentiated Half-Logistic-G (OEHL-G) family proposed by Afify et al. (2017).

In this study, the OEHL-G generator is applied to the Exponentiated Exponential baseline to obtain the Odd Exponentiated Half-Logistic Exponentiated Exponential (OEHL-EE) distribution. The resulting model is a four-parameter distribution designed to provide additional flexibility for modelling positive and skewed observations.

The theoretical development of the OEHL-EE model includes its distributional and reliability properties, estimation procedures and simulation performance. The present study concentrates specifically on its empirical applicability. Four real-life datasets are employed to examine whether the additional

flexibility introduced by the proposed model translates into improved empirical fit.

The four datasets represent different application contexts: economic observations, reliability/lifetime observations, a classical real-data example and engineering failure-time data. The proposed model is compared with several established and recently developed distributions. Model performance is evaluated using likelihood-based information criteria and formal goodness-of-fit statistics.

The remainder of the paper is organized as follows. Section 2 briefly describes the OEHL-EE model. Section 3 presents the datasets and competing distributions. Section 4 describes the estimation and model-assessment procedures. Section 5 presents the

empirical results. Section 6 discusses the findings, while Section 7 concludes the study.

II. THE OEHL-EE DISTRIBUTION

2.1 Definition: Suppose that T is a lifetime random variable with a certain continuous cumulative distribution function (cdf), $G(x; \xi)$. The odds ratio that an object which follows the lifetime T will die at time x is $\frac{G(x; \xi)}{1-G(x; \xi)}$. If the variability of these odds of death is represented by the random variable X and assumed it follows OEHL-G distributions then according to Afify et al. (2017), the cdf of OEHL-G is defined as

$$F(x; \alpha, \lambda, \xi) = \left[\frac{1 - \exp\left(\frac{-\lambda(G(x; \xi))}{1-G(x; \xi)}\right)}{1 + \exp\left(\frac{-\lambda(G(x; \xi))}{1-G(x; \xi)}\right)} \right]^\alpha \quad (1)$$

and the corresponding probability density function (pdf) of the OEHL-G family of distributions is obtained as

$$f(x; \alpha, \lambda, \xi) = \frac{2\alpha\lambda(g(x; \xi)) \left[\exp\left(\frac{-\lambda(G(x; \xi))}{1-G(x; \xi)}\right) \right] \left[1 - \exp\left(\frac{-\lambda(G(x; \xi))}{1-G(x; \xi)}\right) \right]^{\alpha-1}}{[1 - G(x; \xi)]^2 \left[1 + \exp\left(\frac{-\lambda(G(x; \xi))}{1-G(x; \xi)}\right) \right]^{\alpha+1}} \quad (2)$$

where,

$G(x; \xi)$ is the cdf of baseline distribution which depends on a parameter vector ξ ,
 $g(x; \xi)$ is the pdf of baseline distribution which depends on a parameter vector ξ ,
 α is a shape parameter; $\alpha > 0$,
 λ is a shape parameter; $\lambda > 0$,
 $x > 0$.

2.2. Odd Exponentiated Half Logistic Exponentiated Exponential (OEHL-EE) Distribution

Definition: A random variable X is said to follow exponentiated exponential (EE) distribution with parameters β and γ , if its cdf has the following form;

$$G(x) = (1 - \exp(-\beta x))^\gamma \quad (3)$$

The corresponding pdf of the EE distribution is given as

$$g(x) = \gamma\beta(\exp(-\beta x))(1 - \exp(-\beta x))^{\gamma-1} \quad (4)$$

$x > 0; \beta, \gamma > 0$

The EE distribution in equation (3) has shape parameters γ , and scale parameter β . Note that the shape parameter γ represent the exponential family when it is equaled to 1 (Gupta and Kundu, 2001).

Suppose a random variable X follows an OEHL-EE distribution with parameters $\alpha, \lambda, \gamma, \beta$ and using the cdf and pdf of EE distribution as the baseline distribution then the cdf of OEHL-EE distribution is obtained by substituting equation (3) in equation (1); and it is defined

$$F(x; \alpha, \lambda, \gamma, \beta) = \left[\frac{1 - \exp\left(\frac{-\lambda(1 - \exp(-\beta x))^\gamma}{1 - (1 - \exp(-\beta x))^\gamma}\right)}{1 + \exp\left(\frac{-\lambda(1 - \exp(-\beta x))^\gamma}{1 - (1 - \exp(-\beta x))^\gamma}\right)} \right]^\alpha \quad (5)$$

The corresponding pdf of OEHL-EE distribution is obtained by differentiating (5) with respect to x or by substituting equations (3) and (4) in equation (2), we have

$$f(x; \alpha, \lambda, \gamma, \beta) = \frac{2\alpha\lambda(\gamma\beta(\exp(-\beta x))(1 - \exp(-\beta x))^{\gamma-1}) \left[\exp\left(\frac{-\lambda(1 - \exp(-\beta x))^\gamma}{1 - (1 - \exp(-\beta x))^\gamma}\right) \right] \left[1 - \exp\left(\frac{-\lambda(1 - \exp(-\beta x))^\gamma}{1 - (1 - \exp(-\beta x))^\gamma}\right) \right]^{\alpha-1}}{\left[1 - (1 - \exp(-\beta x))^\gamma \right]^2 \left[1 + \exp\left(\frac{-\lambda(1 - \exp(-\beta x))^\gamma}{1 - (1 - \exp(-\beta x))^\gamma}\right) \right]^{\alpha+1}} \quad (6)$$

$$x \in (0, \infty); \alpha, \lambda, \gamma, \beta > 0$$

where, α , λ , and γ are the shape parameters; β is the scale parameter.

2.3 Hazard Function

The hazard function is also known as the failure or risk function and is the probability that a system or an individual or a unit will fail or die for an interval of time.

The hazard function is defined as

$$h(x) = \frac{f(x)}{1 - F(x)} = \frac{f(x)}{S(x)} \quad (7)$$

where, $F(x)$ and $f(x)$ are the cdf and pdf of a baseline (parent) distribution, respectively.

The hazard function of OEHL-EE distribution is given as

$$h(x) = \frac{2\alpha\lambda(\gamma\beta(\exp(-\beta x))(1 - \exp(-\beta x))^{\gamma-1}) \left[\exp\left(\frac{-\lambda(1 - \exp(-\beta x))^\gamma}{1 - (1 - \exp(-\beta x))^\gamma}\right) \right] \left[1 - \exp\left(\frac{-\lambda(1 - \exp(-\beta x))^\gamma}{1 - (1 - \exp(-\beta x))^\gamma}\right) \right]^{\alpha-1}}{\left\{ 1 - \left[\frac{1 - \exp\left(\frac{-\lambda(1 - \exp(-\beta x))^\gamma}{1 - (1 - \exp(-\beta x))^\gamma}\right)}{1 + \exp\left(\frac{-\lambda(1 - \exp(-\beta x))^\gamma}{1 - (1 - \exp(-\beta x))^\gamma}\right)} \right]^\alpha \right\} \left\{ \left[1 - (1 - \exp(-\beta x))^\gamma \right]^2 \left[1 + \exp\left(\frac{-\lambda(1 - \exp(-\beta x))^\gamma}{1 - (1 - \exp(-\beta x))^\gamma}\right) \right]^{\alpha+1} \right\}} \quad (8)$$

2.6 Survival or Reliability Function of OEHL-EE Distribution

The survival function of OEHL-EE distribution is given as

$$S(x) = 1 - \left\{ \frac{1 - \exp\left(\frac{-\lambda(1 - \exp(-\beta x))^\gamma}{1 - (1 - \exp(-\beta x))^\gamma}\right)}{1 + \exp\left(\frac{-\lambda(1 - \exp(-\beta x))^\gamma}{1 - (1 - \exp(-\beta x))^\gamma}\right)} \right\}^\alpha \quad (9)$$

III. REAL-LIFE DATASETS

Two datasets were employed to investigate the empirical performance of the OEHL-EE model.

3.1 Dataset I: Gross and Clark Data

The first dataset is the Gross and Clark dataset. The descriptive statistics reported below:

Table 1: Descriptive statistics for dataset I

Statistic	Min.	Q1	Median	Mean	Q3	Max.	Variance	Skewness	Kurtosis
Dataset I	1.1000	1.4750	1.7000	1.9000	2.0500	4.1000	0.4958	1.5924	2.3465

The positive skewness again indicates substantial departure from symmetry.

3.2 Dataset II: Turbocharger Failure-Time Data

The second dataset represents the time to failure, measured in (10^3) hours, of a turbocharger from one type of engine. The reported descriptive statistics are:

Table 2: Descriptive statistics for dataset II

Statistic	Min.	Q1	Median	Mean	Q3	Max.	Variance	Skewness	Kurtosis
Dataset II	1.6000	5.0750	6.5000	6.2525	7.8250	9.0000	3.8241	-0.6379	-0.4894

Unlike the first three datasets, Dataset IV has negative sample skewness.

IV. COMPETING MODELS

To assess the empirical usefulness of the OEHL-EE distribution, it is fitted alongside several competing probability models.

The competing distributions considered in the thesis include:

Weibull (W);
Gamma (G);
Kumaraswamy-Weibull (Kum-W);
Odd Exponentiated Half-Logistic-Weibull (OEHL-W);
Odd Exponentiated Half-Logistic-Gamma (OEHL-Ga);
Odd Exponentiated Half-Logistic-Normal (OEHL-N);
Exponentiated Exponential (EE);
Transmuted Exponential-Weibull (TE-W);
New Exponential-Exponential (NE-E);
Inverse Power Chi-Square (IPCS); and
Half-Logistic Exponentiated Inverted Weibull (HLEIW).

These models provide a broad comparison involving both conventional lifetime distributions and generalized models related to the OEHL construction.

V. ESTIMATION AND MODEL ASSESSMENT

5.1 Maximum Likelihood Estimation

Let $x_1, x_2, \dots, x_n$ be the random sample of size n from the OEHL-EE distribution with unknown parameters α, λ, γ and β .

Let $\theta = (\alpha, \lambda, \gamma, \beta)^T$ be the $p \times 1$ parameter vector.

The pdf of OEHL-EE distribution is given as

$$f(x; \alpha, \lambda, \gamma, \beta) = \frac{2\alpha\lambda(\gamma\beta(\exp(-\beta x))(1-\exp(-\beta x))^{\gamma-1}) \left[\exp\left(\frac{-\lambda(1-\exp(-\beta x))^{\gamma}}{1-(1-\exp(-\beta x))^{\gamma}}\right) \right] \left[1 - \exp\left(\frac{-\lambda(1-\exp(-\beta x))^{\gamma}}{1-(1-\exp(-\beta x))^{\gamma}}\right) \right]^{\alpha-1}}{\left[1 - (1-\exp(-\beta x))^{\gamma} \right]^2 \left[1 + \exp\left(\frac{-\lambda(1-\exp(-\beta x))^{\gamma}}{1-(1-\exp(-\beta x))^{\gamma}}\right) \right]^{\alpha+1}} \quad (10)$$

$x \in (0, \infty); \alpha, \lambda, \gamma, \beta > 0$

where, α, λ , and γ are the shape parameters; β is the scale parameter.

For determining the MLE of θ using (10)

$$\begin{aligned} L(\theta) &= \prod_{i=1}^n f(x_i; \alpha, \lambda, \gamma, \beta) \\ &= \prod_{i=1}^n \left(\frac{2\alpha\lambda(\gamma\beta e^{-\beta x_i}(1-e^{-\beta x_i})^{\gamma-1}) \left[\exp\left(\frac{-\lambda(1-e^{-\beta x_i})^{\gamma}}{1-(1-e^{-\beta x_i})^{\gamma}}\right) \right] \left[1 - \exp\left(\frac{-\lambda(1-e^{-\beta x_i})^{\gamma}}{1-(1-e^{-\beta x_i})^{\gamma}}\right) \right]^{\alpha-1}}{\left[1 - (1-e^{-\beta x_i})^{\gamma} \right]^2 \left[1 + \exp\left(\frac{-\lambda(1-e^{-\beta x_i})^{\gamma}}{1-(1-e^{-\beta x_i})^{\gamma}}\right) \right]^{\alpha+1}} \right) \\ &= \frac{2^n \alpha^n \lambda^n \gamma^n \beta^n \prod_{i=1}^n e^{-\beta x_i} \prod_{i=1}^n (1-e^{-\beta x_i})^{\gamma-1} \prod_{i=1}^n \left[\exp\left(\frac{-\lambda(1-e^{-\beta x_i})^{\gamma}}{1-(1-e^{-\beta x_i})^{\gamma}}\right) \right] \prod_{i=1}^n \left[1 - \exp\left(\frac{-\lambda(1-e^{-\beta x_i})^{\gamma}}{1-(1-e^{-\beta x_i})^{\gamma}}\right) \right]^{\alpha-1}}{\prod_{i=1}^n \left[1 - (1-e^{-\beta x_i})^{\gamma} \right]^2 \prod_{i=1}^n \left[1 + \exp\left(\frac{-\lambda(1-e^{-\beta x_i})^{\gamma}}{1-(1-e^{-\beta x_i})^{\gamma}}\right) \right]^{\alpha+1}} \quad (11) \end{aligned}$$

Taking the log-likelihood of (11), i.e., $\log L(\theta)$, we have

$$\begin{aligned} \log L(\theta) &= \log \left(\frac{2^n \alpha^n \lambda^n \gamma^n \beta^n \prod_{i=1}^n e^{-\beta x_i} \prod_{i=1}^n (1-e^{-\beta x_i})^{\gamma-1} \prod_{i=1}^n \left[\exp\left(\frac{-\lambda(1-e^{-\beta x_i})^{\gamma}}{1-(1-e^{-\beta x_i})^{\gamma}}\right) \right] \prod_{i=1}^n \left[1 - \exp\left(\frac{-\lambda(1-e^{-\beta x_i})^{\gamma}}{1-(1-e^{-\beta x_i})^{\gamma}}\right) \right]^{\alpha-1}}{\prod_{i=1}^n \left[1 - (1-e^{-\beta x_i})^{\gamma} \right]^2 \prod_{i=1}^n \left[1 + \exp\left(\frac{-\lambda(1-e^{-\beta x_i})^{\gamma}}{1-(1-e^{-\beta x_i})^{\gamma}}\right) \right]^{\alpha+1}} \right) \quad (12) \end{aligned}$$

$$\begin{aligned}
 &= n \log 2 + n \log \alpha + n \log \lambda + n \log \gamma + n \log \beta - \beta \sum_{i=1}^n x_i + (\gamma - 1) \sum_{i=1}^n \log(1 - e^{-\beta x_i}) \\
 &\quad + \sum_{i=1}^n \left(\frac{-\lambda(1 - e^{-\beta x_i})^\gamma}{1 - (1 - e^{-\beta x_i})^\gamma} \right) + (\alpha - 1) \sum_{i=1}^n \log \left(1 - \exp \left(\frac{-\lambda(1 - e^{-\beta x_i})^\gamma}{1 - (1 - e^{-\beta x_i})^\gamma} \right) \right) \\
 &\quad - 2 \sum_{i=1}^n \log(1 - (1 - e^{-\beta x_i})^\gamma) - (\alpha + 1) \sum_{i=1}^n \log \left(1 + \exp \left(\frac{-\lambda(1 - e^{-\beta x_i})^\gamma}{1 - (1 - e^{-\beta x_i})^\gamma} \right) \right)
 \end{aligned} \tag{13}$$

Differentiating (39) with respect to each parameter α, λ, γ , and β , and equating each of the derivative to zero

$$\frac{d \log L(\theta)}{d \alpha} = \frac{n}{\alpha} + \sum_{i=1}^n \log \left(1 - \exp \left(\frac{-\lambda(1 - e^{-\beta x_i})^\gamma}{1 - (1 - e^{-\beta x_i})^\gamma} \right) \right) - \sum_{i=1}^n \log \left(1 + \exp \left(\frac{-\lambda(1 - e^{-\beta x_i})^\gamma}{1 - (1 - e^{-\beta x_i})^\gamma} \right) \right) = 0 \tag{14}$$

$$\begin{aligned}
 \frac{d \log L(\theta)}{d \lambda} &= \frac{n}{\lambda} - \sum_{i=1}^n \frac{(1 - e^{-\beta x_i})^\gamma}{(1 - (1 - e^{-\beta x_i})^\gamma)^2} \left\{ \exp \left(\frac{-\lambda(1 - e^{-\beta x_i})^\gamma}{1 - (1 - e^{-\beta x_i})^\gamma} \right) - \frac{(\alpha - 1) \exp \left(\frac{-\lambda(1 - e^{-\beta x_i})^\gamma}{1 - (1 - e^{-\beta x_i})^\gamma} \right)}{1 - \exp \left(\frac{-\lambda(1 - e^{-\beta x_i})^\gamma}{1 - (1 - e^{-\beta x_i})^\gamma} \right)} \right. \\
 &\quad \left. - \frac{(\alpha + 1)}{1 + \exp \left(\frac{-\lambda(1 - e^{-\beta x_i})^\gamma}{1 - (1 - e^{-\beta x_i})^\gamma} \right)} \right\} = 0
 \end{aligned} \tag{15}$$

$$\begin{aligned}
 \frac{d \log L(\theta)}{d \gamma} &= \frac{n}{\gamma} + \sum_{i=1}^n \log(1 - e^{-\beta x_i}) - 2 \sum_{i=1}^n \frac{(1 - e^{-\beta x_i})^\gamma \log(1 - e^{-\beta x_i})}{1 - (1 - e^{-\beta x_i})^\gamma} \\
 &\quad - \sum_{i=1}^n \frac{\lambda(1 - e^{-\beta x_i})^\gamma \log(1 - e^{-\beta x_i})}{1 - (1 - e^{-\beta x_i})^\gamma} \left\{ \exp \left(\frac{-\lambda(1 - e^{-\beta x_i})^\gamma}{1 - (1 - e^{-\beta x_i})^\gamma} \right) \right. \\
 &\quad \left. - \frac{(\alpha - 1) \exp \left(\frac{-\lambda(1 - e^{-\beta x_i})^\gamma}{1 - (1 - e^{-\beta x_i})^\gamma} \right)}{1 - \exp \left(\frac{-\lambda(1 - e^{-\beta x_i})^\gamma}{1 - (1 - e^{-\beta x_i})^\gamma} \right)} - \frac{(\alpha + 1)}{1 + \exp \left(\frac{-\lambda(1 - e^{-\beta x_i})^\gamma}{1 - (1 - e^{-\beta x_i})^\gamma} \right)} \right\} = 0
 \end{aligned} \tag{16}$$

$$\frac{d \log L(\theta)}{d \beta} = \frac{n}{\beta} - \sum_{i=1}^n x_i + (\gamma - 1) \sum_{i=1}^n \frac{e^{-\beta x_i} x_i}{1 - e^{-\beta x_i}} - 2 \sum_{i=1}^n \frac{\gamma e^{-\beta x_i} x_i (1 - e^{-\beta x_i})^{\gamma-1}}{1 - (1 - e^{-\beta x_i})^\gamma} = 0 \tag{17}$$

By setting the nonlinear system of equations to zero, that is, $\frac{d \log L(\theta)}{d \alpha} = \frac{d \log L(\theta)}{d \lambda} = \frac{d \log L(\theta)}{d \gamma} = \frac{d \log L(\theta)}{d \beta} = 0$ and solving them simultaneously gives the MLE $\theta = (\alpha, \lambda, \gamma, \beta)^T$. Moreover, numerical computations are needed to find the

solution of this nonlinear system of equations, $\frac{d\log L(\theta)}{d\alpha} = \frac{d\log L(\theta)}{d\lambda} = \frac{d\log L(\theta)}{d\gamma} = \frac{d\log L(\theta)}{d\beta} = 0$ and it is usually more convenient to obtain by using nonlinear optimization methods such as the Newton-Rapson algorithm.

5.2 Information Criteria

Five likelihood-based criteria are considered:

(i) Log- Likelihood (**LL**)

(ii) Akaike Information Criterion (AIC)

$$AIC = -2LL + 2k$$

(iii) Consistent Akaike Information Criterion (CAIC)

$$CAIC = -2LL + \frac{2kn}{(n-k-1)}$$

(iv) Hannan-Quinn Information Criteria (HQIC)

$$HQIC = -2LL + 2k\log(\log(n))$$

(v) Bayesian Information Criterion (BIC)

$$BIC$$

$$= -LL + k\log(n)$$

where LL is the log-likelihood function, k is the number of model parameters and n is the size of the sample.

VI. EMPIRICAL RESULTS

6.1 Results for Dataset I

For the Gross and Clark dataset, the OEHL-EE model produced:

Table 3: Estimated Parameters and Goodness-of-Fit for Dataset I (Gross and Clark)

Model	ML Estimates	-LL	AIC	CAIC	HQIC	BIC	W^*	A^*	D^*
OEHL-EE	$\hat{\alpha} = 508.224$	15.7892	39.5785	32.1118	40.3560	35.9672	0.0342	0.2269	0.1161
	$\hat{\lambda} = 19.9999$								
	$\hat{\gamma} = 0.4160$								
	$\hat{\beta} = 0.0244$								
OEHL-W	$\hat{\theta} = 100.0$	15.8554	39.7107	32.2440	40.4882	36.0995	0.0370	0.2294	0.1208
	$\hat{\lambda} = 5144.06$								
	$\hat{a} = 0.7031$								
	$\hat{b} = 28416.15$								
OEHL-Ga	$\hat{\theta} = 20.0$	16.4977	40.9953	33.5287	41.7728	37.3841	0.0596	0.3559	0.1452
	$\hat{\lambda} = 20.0$								
	$\hat{a} = 0.8278$								
	$\hat{b} = 15.6200$								
OEHL-N	$\hat{\alpha} = 20.0$	19.2859	46.5718	39.1052	47.3493	42.9606	0.1445	0.8487	0.1946
	$\hat{\lambda} = 20.0$								
	$\hat{\mu} = 5.6360$								
	$\hat{\sigma} = 3.9248$								
EE	$\hat{\gamma} = 36.6833$	16.2606	36.5212	32.7565	36.91001	34.7156	0.0477	0.3105	0.1343
	$\hat{\beta} = 2.2352$								
Kum-W	$\hat{a} = 3.9798$	17.7134	43.4268	35.9602	44.2043	39.8156	0.1051	0.6015	0.1722
	$\hat{b} = 0.4515$								
	$\hat{c} = 2.0205$								
	$\hat{\lambda} = 0.9119$								
W	$\hat{\beta} = 2.7870$	20.5864	45.1728	41.4081	45.5616	43.3672	0.1834	1.0835	0.1850
	$\hat{\eta} = 2.1300$								
G	$\hat{\alpha} = 9.6694$	17.8186	39.6372	35.8725	40.0260	37.8316	0.1025	0.5990	0.1734
	$\hat{\beta} = 5.0892$								

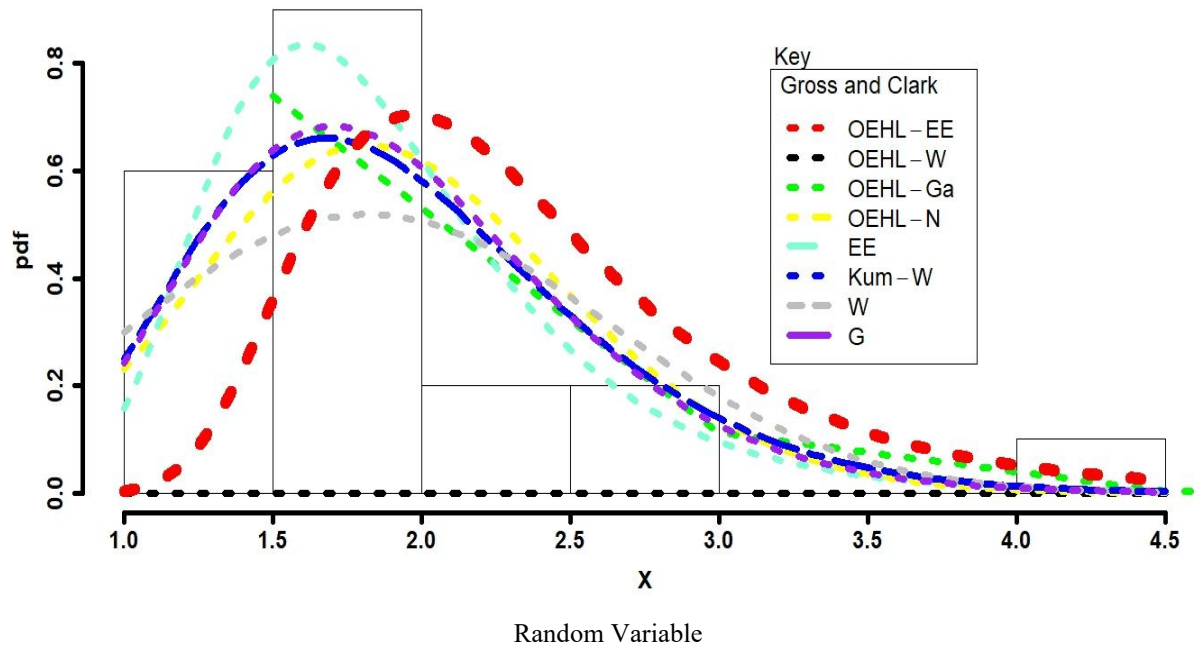


Figure 1: Plot of PDF for the estimates of the fitted models to dataset I (Gross and Clark dataset)

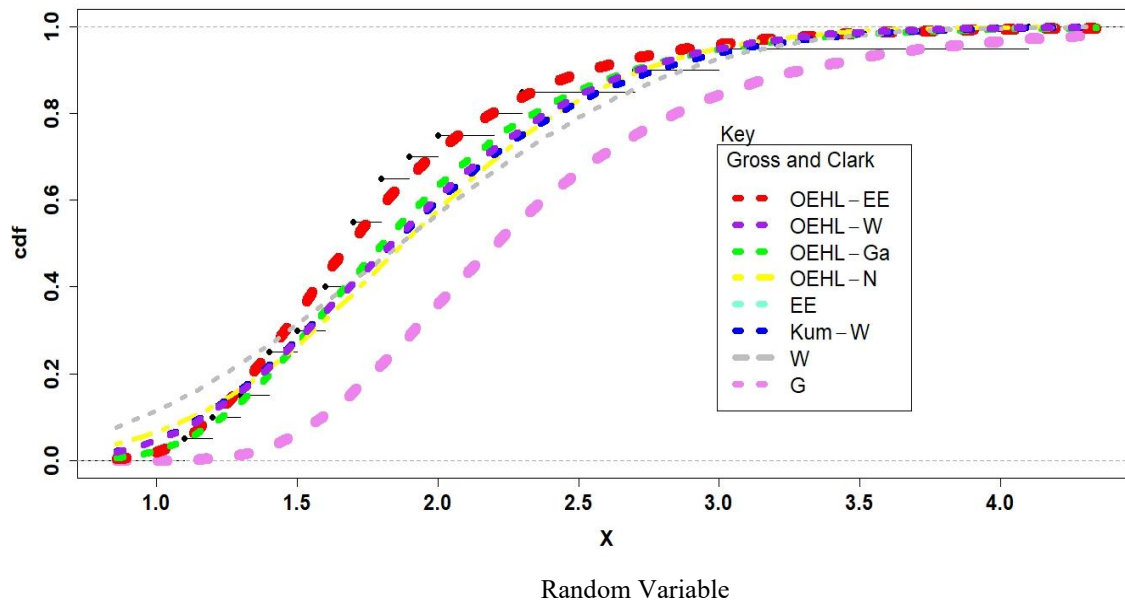


Figure 2: Plot of CDF of the fitted models and empirical CDF to dataset I (Gross and Clark dataset)

6.2 Results for Dataset II

For the turbocharger failure-time data, the OEHL-EE model produced:

Table 4: Estimated Parameters and Goodness-of-Fit for Dataset II (the time to failure (10^3h) of turbocharger of one type of engine)

Model	ML Estimates	-LL	AIC	CAIC	HQIC	BIC	W^*	A^*	D^*
OEHL-EE	$\hat{\alpha}$								
	= 0.3401								
	$\hat{\lambda}$								
	= 0.0048								
	$\hat{\gamma}$								
	= 40.5722	77.9097	163.8194	156.0480	166.2620	161.0407	0.0226	0.1389	0.0648
	$\hat{\beta}$								
	= 1.1476								
OEHL-W	$\hat{\theta} = 0.2557$								
	$\hat{\lambda}$								
	= 233601.8								
	$\hat{a} = 9.9387$	78.4230	164.8460	157.0746	167.2886	162.0673	0.0186	0.1392	0.0684
	$\hat{b}$								
	= 27.8962								
OEHL-Ga	$\hat{\theta} = 0.3530$								
	$\hat{\lambda} = 0.0572$								
	$\hat{a}$	78.1270	164.2540	156.4826	166.6966	161.4753	0.0232	0.1374	0.0679
	= 13.7119								
	$\hat{b} = 0.3920$								
OEHL-N	$\hat{\alpha} = 0.1813$								
	$\hat{\lambda} = 0.1902$								
	$\hat{\mu} = 7.4079$	78.0208	164.0415	156.2701	166.4841	161.2628	0.0118	0.0969	0.0472
	$\hat{\sigma} = 0.9697$								
EE	$\hat{\gamma} = 9.5148$	90.1427	184.2853	180.3934	185.5066	182.8960	0.2801	1.7478	0.1542
	$\hat{\beta} = 0.4498$								
Kum-W	$\hat{a} = 1.4245$								
	$\hat{b} = 1.4034$								
	$\hat{c} = 2.8527$	83.6387	175.2773	167.5059	177.7199	172.4986	0.1142	0.8046	0.1125
	$\hat{\lambda} = 0.1447$								
HLEIW	$\hat{\alpha}$								
	= 14.0218	87.8438	181.6877	175.8544	183.5196	179.6037	0.2091	1.3915	0.1273
	$\hat{\theta} = 0.8750$								
	$\hat{\rho} = 20.0$								
W	$\hat{\beta} = 3.8725$	82.4755	168.9510	165.0591	170.1723	167.5617	0.0815	0.6584	0.1077
	$\hat{\eta} = 6.9200$								
G	$\hat{\alpha} = 7.7226$	87.4103	178.8205	174.9286	180.0418	177.4312	0.2073	1.3561	0.1278
	$\hat{\beta} = 1.2351$								
NE-E	$\hat{\lambda} = 0.3999$	113.3193	228.6385	226.6911	229.2492	227.9438	1.7370	8.6005	0.3631

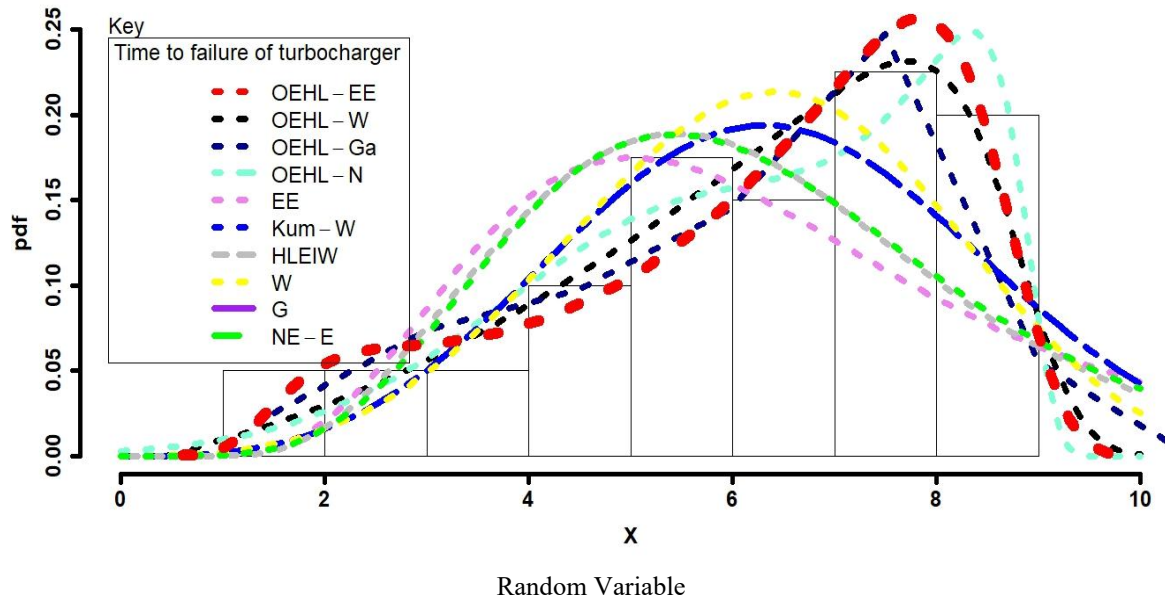


Figure 3: Plot of PDF for the estimates of the fitted models to dataset II (the time to failure (10^3h) of turbocharger of one type of engine dataset)

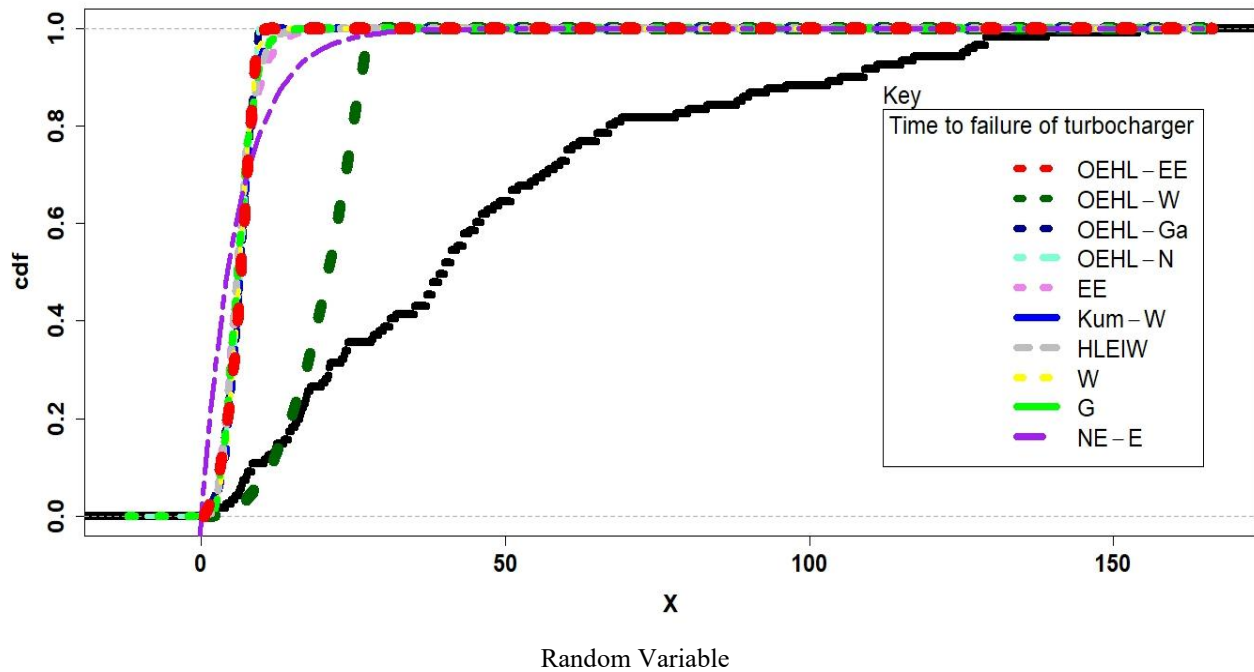


Figure 4: Plot of CDF of the fitted models and empirical CDF to dataset II (the time to failure (10^3h) of turbocharger of one type of engine dataset)

VII. DISCUSSION

The empirical analysis demonstrates that the OEHL-EE model can be applied to datasets from substantially different contexts. The datasets include an economic time-related dataset, a fatigue-fracture lifetime dataset,

a classical statistical dataset and an engineering failure-time dataset.

The descriptive statistics indicate that the first dataset was positively skewed, while Dataset II exhibits negative sample skewness. The range of distributional

characteristics across these datasets provides an opportunity to examine the flexibility of the proposed model under different empirical shapes.

The paper further reports that the fitted density and cumulative distribution plots provide close agreement between the OEHL-EE model and the observed empirical distributions. The corresponding figures are available for each of the two datasets.

An important empirical feature is that the OEHL-EE model is not evaluated solely against the classical Weibull, Gamma and Exponentiated Exponential distributions. The comparison also includes several generalized models, including OEHL-based competitors. This provides a more demanding assessment of the proposed model's empirical usefulness.

The results therefore provide empirical evidence that the additional flexibility introduced through the OEHL transformation can be useful when modelling positive and skewed observations. However, model selection should be interpreted in conjunction with the characteristics of the particular dataset, the parameter estimates and graphical diagnostics rather than relying on a single goodness-of-fit statistic

VII. CONCLUSION

This study examined the empirical performance of the Odd Exponentiated Half-Logistic Exponentiated Exponential distribution using four real-life datasets. The applications covered Gross and Clark dataset, and turbocharger failure times.

The OEHL-EE model was estimated using maximum likelihood and compared with a range of conventional and generalized probability distributions. The assessment employed negative log-likelihood, AIC, CAIC, HQIC, BIC, Anderson–Darling, Cramér–von Mises and Kolmogorov–Smirnov statistics, together with graphical comparisons of fitted densities and cumulative distribution functions.

The reported empirical results show that OEHL-EE provides a close fit to the two datasets and performs competitively against the alternative models

considered. Its application to datasets with different degrees and directions of skewness illustrates the flexibility provided by its additional shape parameters. The findings support the use of OEHL-EE as a candidate model for positive-valued lifetime, reliability and related datasets. Future research may investigate its performance with additional datasets from reliability engineering, survival analysis, finance, environmental studies and other fields where flexible continuous probability models are required.

DECLARATIONS

Data Availability

The empirical datasets used in this study are described in the original thesis and were obtained from the sources identified therein. The Kevlar 373/epoxy dataset consists of 76 complete observations and has previously been used in published statistical studies.

Conflict of Interest

The author declares that there is no conflict of interest.

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Author Contributions

The author conceived the study, developed the statistical methodology, conducted the analysis and prepared the manuscript.

REFERENCES

- [1] Z. Afify, C. Lee, and A. Alzaatreh, "Odd exponentiated half logistic distribution: Properties and applications," *Journal of Statistical Theory and Practice*, vol. 11, no. 3, pp. 417–432, 2017.
- [2] G. M. Cordeiro and M. de Castro, "A new family of generalized distributions," *Journal of Statistical Computation and Simulation*, vol. 81, no. 7, pp. 883–898, 2011. [Taylor & Francis](#)
- [3] R. D. Gupta and D. Kundu, "Exponentiated exponential family: An alternative to gamma and Weibull distributions," *Biometrical Journal*, vol. 43, no. 1, pp. 117–130, 2001. [Wiley](#)
- [4] A. I. Ishaq, A. A. Abiodun, A. A. Suleiman, A. Usman, A. S. Mohammed, and M. Tasiu,

- “Modelling Nigerian inflation rates from January 2003 to June 2023 using newly developed inverse power chi-square distribution,” in *Proceedings of the 4th International Conference on Data Analytics for Business and Industry*, 2023. [IEEE](#)
- [5] K. Jayakumar and M. G. Babu, “T-transmuted X family of distributions,” *Statistica*, vol. 77, no. 3, pp. 261–276, 2017.
- [6] A. S. Mohammed and F. I. Ugwuowo, “A new four-parameter Weibull distribution with application to failure time data,” *FUDMA Journal of Sciences*, vol. 4, no. 3, pp. 563–575, 2020. [FUDMA Journal of Sciences](#)
- [7] O. D. Ogunwale, T. M. Ayeni, F. A. Osunronbi, R. A. Adeleke, and E. A. Odukoya, “The new Exponential–Exponential distribution: Theory and properties,” *Quest Journal of Research in Applied Mathematics*, vol. 8, no. 8, pp. 13–19, 2022.
- [8] O. I. Oseghale, V. E. Laoye, and Z. A. Adeleke, “Half logistic exponentiated inverted Weibull distribution: Properties and application to survival data,” *International Journal of Applied Science and Mathematical Theory*, vol. 11, no. 2, pp. 100–110, 2025. [IIARD Journals](#)
- [9] E. A. Owoloko, P. E. Oguntunde, and A. O. Adejumo, “Performance rating of the transmuted exponential distribution: An analytical approach,” *SpringerPlus*, vol. 4, Art. no. 818, 2015. [Springer](#)
- [10] Oluyede, T. Moakofi, and G. Lekono, “The new Marshall–Olkin–type II exponentiated half-logistic–odd Burr X-G family of distributions with properties and applications,” *Stats*, vol. 8, no. 2, Art. no. 26, 2025. [MDPI](#)
- [11] E. W. Stacy, “A generalization of the gamma distribution,” *Annals of Mathematical Statistics*, vol. 33, no. 3, pp. 1187–1192, 1962. [Project Euclid](#)
- [12] W. Weibull, “A statistical distribution function of wide applicability,” *Journal of Applied Mechanics*, vol. 18, no. 3, pp. 293–297, 1951. [ASME](#)