

Oscillation Criteria for Solutions of First Order Advanced Differential Equations with Monotone Arguments

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Abstract- This study considered the oscillation criteria for solutions of first order advanced differential equations with monotone argument the condition for oscillation of ADEs with monotone argument were established. We employed the method of Chatzarakis and Li to improve the first order ADEs to monotone argument. The result obtained concludes that the oscillation of first order first order advanced is also oscillatory when the value of τ is large enough. Therefore, it is recommended for further modification on our criteria so that it can accommodate higher order delay and advanced differential equation.

Keyword: Advanced Differential Equation, Criteria, Chatzarakis and Li, Differential Equation

I. INTRODUCTION

Ordinary differential equations (ODEs) are used to describe many phenomena of physical interest, because of their frequent use in the investigation of a wide variety of problems in the field of science, such as transforming mechanical problems to system of differential equations in engineering, principles in physics, fluid dynamics such as measuring the rate of flow or concentration of impurities in a fluid tank, in Industrial Chemistry and modeling the dynamics of population. ODE contains derivatives of which depend on the solution at the present value of the independent variable that is time (Shampine & Thampson, [1]).

Delay differential equations (DDEs) are a type of differential equations where the derivative of the unknown function, at a certain time is given in terms of the values of the function, at previous times. DDEs are also referred in the literature as time-delay systems, systems with after-effect or dead-time, hereditary systems or equations with delay arguments, or differential- difference equations. They belong to class of systems with the functional state, that is partial differential equations (PDEs) which are infinite dimensional as opposed to ordinary differential

equation having finite dimensional state vector (Chatzarakis & Li, [2]).

Advanced differential equations (ADE) are a type of differential equations where the derivative of the unknown function, at a certain time, is given in terms of the values of the function, at future times (Chatzarakis & Li, [3]).

The equations below represent the general forms of DDEs and ADEs with delay arguments respectively.

$$x^{(n)}(t) + p(t) x(\tau(t)) = 0 \quad (1)$$

where $p(t)$, $\tau(t) < t$, $t \geq t_0$ is nonnegative integer

$$x^{(n)}(t) - q(t) x(\sigma(t)) = 0 \quad (2)$$

where $q(t)$, $\sigma > t$, $t \geq t_0$ is nonnegative integer

The first derivative of equation (1) gives DDE with an argument, which can be expressed in the form of

$$x'(t) + p(t) x(\tau(t)) = 0 \quad (3)$$

where $p(t)$ is nonnegative integer and $\tau(t)$ is monotone function with $\tau(t) < t$, $0 < t < \infty$.

Similarly the first derivative of equation (2) can be expressed as

$$x'(t) - q(t) x(\sigma(t)) = 0 \quad (4)$$

where $q(t)$ is nonnegative integer and $\sigma(t)$ is monotone function with $\sigma(t) > t$, $0 < t < \infty$.

In this study we considered the oscillation of equation (2) when the argument is monotone. The solution of equation (2) is said to be oscillatory if its solutions has arbitrarily large zeros on the half range, otherwise is non-oscillatory.

Chatzarakis and Ocalan [4] considered first order differential equation with non-monotone retard argument in equation (4) where $p(t) \geq 0$ and $\tau(t)$ if

a function of positive real numbers such that $\tau(t) \leq t$ for $t \geq t_0$, and $\lim_{t \rightarrow \infty} \tau(t) = \infty$.

Under the assumption that the retarded argument $\tau(t)$ is non-monotone, it's used in established the new oscillation criterion that involving $\liminf$ and oscillation condition

$$\liminf_{t \rightarrow \infty} \int_{\tau(t)}^t p(s) ds > \frac{1}{e} \quad (5)$$

is not satisfied.

They considered non-monotone argument when solving differential equation (5) rather than using pure mathematics approach. Then the monotone argument almost always becomes non-monotone and also prove that every solution of (4) oscillate if

$$\limsup_{t \rightarrow \infty} [t - \tau(t)] < \infty \quad \text{and} \quad \liminf_{t \rightarrow \infty} [t - \tau(t)] \lim_{t \rightarrow \infty} p(t) > \frac{1}{e}. \quad (6)$$

El-Morshedy and Attia [5] investigated the oscillation criterion for delay differential equation with non-monotone arguments of the form (6). They obtained new oscillation criterion of the $\limsup$ type also constructed example to showed the applicability and the strength of the obtained over existing ones.

Lakshmikanham [6] studied the oscillatory behavior of the first order differential equation with deviating arguments of the general form

$$x'(t) + \sum_j^n p_i x(t - \tau_i) = 0, \quad (7)$$

where $p_i, \tau_i, 1 \leq i \leq n$ are constants. The necessary and sufficient conditions have been obtained guarantee that equation (7) is oscillatory and he also established the equivalence of two kinds of oscillation for equation (7) is oscillatory if and the characteristic equation

$$\lambda + \sum_i^n p_i e^{-\pi \tau_i} = 0$$

has no real roots.

Moremedi and Stavrouslakis [7] considered the first-order linear with several arguments of the form

$$x'(t) + \sum_{j=1}^m p_i(t) x(\tau_i(t)) = 0, \quad t \geq t_0 \quad (8)$$

where the function $p_i, \tau_i \in ([t_0, \infty), R^+)$ for every $i = 1, 2, \dots, m, \tau_i(t) \leq t$ for $t \geq t_0$. They presented an interesting oscillation condition of equation. (8) some example illustrating the results is given.

Chatzarakis [8] presented a sufficient condition involving $\limsup$ in an explicit iterative form implied the oscillation of all solution of differential equation of the form in equation (2.30) with variable non-monotone argument and non-negative coefficient his condition improved all the known results. He also studied the differential equation of the advanced type. Braverman and Karpuz [9] worked on oscillation of differential equations with non-monotone delays of the form

$$x'(t) + p(t)x(h(t)) = 0, \quad p \geq 0, \quad h(t) \geq 0, \quad h(t) \leq t, \quad \lim_{t \rightarrow \infty} h(t) = \infty \quad (9)$$

They demonstrate that the inequality

$$\limsup_{t \rightarrow \infty} \int_{h(t)}^t p(u) du > 1 \quad \text{is not sufficient for}$$

oscillation. Moreover, for any $A > 0$ the relation

$$\limsup_{t \rightarrow \infty} \int_{h(t)}^t p(u) du > A, \quad \text{generally does not imply}$$

oscillation. They also work on similar result for the difference equations in terms of the maximal argument, new sufficient oscillation conditions are presented for differential and difference equation.

Akca, Chatzarakis and stavrouslakis [10] studied oscillation criterion for DDEs with several non-monotone arguments: the oscillatory behaviors of the solution with several non-monotone delay arguments and non-negative coefficient of the form

$$x'(t) + \sum_{i=1}^m p_i(t) x(\tau_i(t)) = 0 \quad \forall t \geq 0 \quad (10)$$

where $p_i, 1 \leq i \leq m$ are function of nonnegative real number, and $\tau_i, 1 \leq i \leq m$ are non-monotone functions of positive real number such that $\tau_i(t) < t$ $t \geq 0$ and $\lim_{t \rightarrow \infty} \tau_i(t) = \infty$ $1 \leq i \leq m$

A sufficient oscillation condition, of equation involving $\limsup$ is obtained and concluded that its oscillatory if all is solution oscillate.

Attia and El-Matary [11] proposed oscillation of a first-order differential equation with several non-monotone delays. They also extend the work of Kwong [12] and Sficas and Stavroulakis [13] for equation with several variable delays. Their result improved and generalized a large number of the existing results.

A sufficient condition, involving $\limsup$ and $\liminf$, for the oscillation of all solution of differential equation with several not necessarily monotone deviating arguments and non-negative coefficient and advanced type are studied. Illustrated the results and the improvement over other known oscillation criteria by examples, numerically solved in MATLAB (Chatzakarakis and Li, [14])

Stavroulaskis, Zhunussova and Dosmagulova [15] considered the first-order linear differential equation with several retarded arguments

$$x'(t) + \sum_{i=1}^m p_i(t)x(\tau_i(t)) = 0, \quad t \geq t_0 \quad (11)$$

where the function $p_i, \tau_i \in C([t_0, \infty), R^+)$ for every $i = 1, 2, \dots, m$, $\tau_i(t) \leq t$ for $t \geq t_0$ and $\lim_{t \rightarrow \infty} \tau_i(t) = \infty$ the established sufficient conditions

under which all solution oscillate. Garab and Stavroulakis [16] considered the first linear differential equation with several delays of the form

$$x'(t) + \sum_{i=1}^m p_i(t)x(t - \tau_i(t)) = 0, \quad t \geq t_0 \quad (12)$$

where $p_i : [t_0, \infty) \rightarrow [0, \infty)$ and $\tau_i : [t_0, \infty) \rightarrow (0, \infty)$ are continuous function such that $t - \tau_i(t) \rightarrow \infty$ as $t \rightarrow \infty$ for all $1 \leq i \leq m$

Sufficient condition are established under which some $\liminf$ -type conditions for oscillation of all solution remain valid if one replaces the $\liminf$ by $\limsup$

. They improved and generalized many existing results in the literature, which is illustrated via examples.

II. METHODOLOGY

New set of criteria for oscillation for first order ADEs with monotone argument

Consider equation (4) with a monotone argument where

$q(t) \in C([t_0, \infty))$ and is nonnegative coefficient and $\sigma(t)$ are monotone argument. If

$$\liminf_{t \rightarrow \infty} \int_t^{\sigma(t)} q(s)ds < \frac{1}{e} \quad (13)$$

and also

$$\limsup_{t \rightarrow \infty} \int_t^{\sigma(t)} q(s)ds < 1 \quad (14)$$

Then equation (14) are oscillatory if

$$h(t) = \sup_{0 < s < \frac{1}{e}} \sigma(t) \quad (15)$$

where $h(t)$ is monotone function. If

$$h(t) \in \left(0, \frac{1}{e}\right)$$

then there exist a small constant $\varepsilon \geq h(t)$ then

$$\liminf_{t \rightarrow \infty} \int_t^{\sigma(t)} q(s)ds = h. \quad (16)$$

and

$$\limsup_{t \rightarrow \infty} \int_t^{\sigma(t)} p(s)ds \geq \varepsilon \quad (17)$$

Integrating equation (4) from t to $\sigma(t)$ to have

$$x(\sigma(t)) = x(t) + \int_t^{\sigma(t)} q(s)x(\sigma(t))ds \quad (18)$$

Let $0 < \lambda < \lambda_1$, then the function

$$\varphi(t) = x(t) \exp \left(\lambda \int_{t_0}^t q(s)ds \right), \quad t \geq t_1 \quad (19)$$

$$x(\sigma(t)) - x(t) + \int_t^{\sigma(t)} q(s)x(\sigma(s))ds = 0 \quad (20)$$

$$x(\sigma(t)) = \int_t^{\sigma(t)} q(s)x(\sigma(s))ds + x(t) \quad (21)$$

Dividing equation (21) by $x(t)$ and considering the left hand side we have

$$\frac{x(\sigma(t))}{x(t)} \geq \lambda \quad (22)$$

Applying $\liminf$ on equation (20) we have

$$\liminf_{t \rightarrow \infty} \frac{x(\sigma(t))}{x(t)} \geq \lambda \quad (23)$$

where λ is the smaller root of equation

$$\lambda = e^{\beta\lambda} \text{ for all } \lambda > 0. \quad (24)$$

By assuming $p(t) = 1$

where

$$f(\lambda) = \lambda - e^{\beta\lambda} \quad (25)$$

III. RESULTS

Establish Results for First Order ADEs with Monotone Argument

Theorem

Suppose that $\sigma(t)$ is a monotone function and $x(t)$ is a positive solution of equation (4) and

$$\lim_{t \rightarrow 0} \frac{x(\sigma(t))}{x(t)} \geq \int_M^N q(t) \int_t^{\sigma(t)} q(s)dt ds \quad (26)$$

Then all the solution of equation (26) oscillates.

Proof

Applying integration on (4) and rearranging we have

$$x(\sigma(t)) = \int_t^{\sigma(t)} q(s)x(\sigma(s))ds + x(t) \quad (27)$$

Dividing equation (25) by we have

$$\frac{x(\sigma(t))}{x(t)} = \int_t^{\sigma(t)} q(s) \frac{x(\sigma(s))}{x(s)} ds + 1 \quad (26)$$

Considering equation (20) then the leftside (26) becomes

$$\frac{x(\sigma(t))}{x(t)} = \lambda \quad (27)$$

By (3.2) equation (27) becomes

$$\liminf_{t \rightarrow \infty} \frac{x(\sigma(t))}{x(t)} > \lambda(t) \quad (28)$$

where $\lambda(t)$ is the smaller root of the equation

$$\lambda(t) = e^{a\lambda(t)} \quad (29)$$

Let $\lambda(t)$ be the characteristic equation and $x(t)$ be positive solution of equation (29)

Now integrating the coefficient $q(t)$ in equation (4)

from t to $\sigma(t)$ we have

$$I(t) = \int_t^{\sigma(t)} q(s)ds. \quad (30)$$

Thus, dividing equation (19) by $x(t)$ we have

$$\frac{\varphi(t)}{x(t)} = q(t) \exp\left(\int_t^{\sigma(t)} \lambda(ds)\right) \quad (31)$$

Combining equation (30) and (31) we have

$$\lambda(t) = q(t) \exp\left(\int_t^{\sigma(t)} q(s)ds \times \frac{1}{I} \int_t^{\sigma(t)} \lambda(s)ds\right) \quad (32)$$

Evaluating equation (32)

$$\lambda(t) = q(t) \exp\left(\int_t^{\sigma(t)} \lambda(s)ds\right) \quad (33)$$

Evaluating equation (33) we have

$$e^{\beta t} \geq t + \frac{\ln(et)}{t} \text{ for } t > 0 \quad (34)$$

Now

Substituting (34) into equation (32) we have

$$\lambda(t) \geq q(t) \left(\frac{1}{I(t)} \int_t^{\sigma(t)} \lambda(s)ds + \frac{\ln(eI(t))}{I(t)} \right) \quad (35)$$

From equation (35) we have

$$\lambda(t)I(t) - q(t) \int_t^{\sigma(t)} \lambda(s)ds \geq q(t) \ln(eI(t)) \quad (36)$$

$$\lambda(t) \int_t^{\sigma(t)} q(s)ds - q(t) \int_t^{\sigma(t)} \lambda(s)ds \geq q(t) \ln\left(e \int_t^{\sigma(t)} q(s)ds\right) \quad (37)$$

Integrating both sides of inequalities (37) from M to N we have

$$\int_M^N \lambda(t) \int_t^{\sigma(t)} q(s)ds dt - \int_M^N q(t) \int_t^{\sigma(t)} \lambda(s)ds dt \geq \int_M^N q(t) \ln\left(e \int_t^{\sigma(t)} q(s)ds\right) dt \quad (38)$$

From equation (38) it follows that

$$\int_M^N \lambda(t) \int_t^{\sigma(t)} q(s)ds dt \geq \int_M^N q(t) \ln\left(e \int_t^{\sigma(t)} p(s)ds\right) dt \quad (39)$$

By using condition (iii) and equation (39) we have

$$\int_M^N \lambda(t) dt \geq \int_M^N q(t) \int_t^{\sigma(t)} q(s) ds dt \quad (40)$$

Applying equation (3.42) on (40) we

$$\lim_{t \rightarrow \infty} \frac{x(\sigma(t))}{x(t)} \geq \int_M^N q(t) \int_t^{\sigma(t)} q(s) ds dt \quad (41)$$

The proof is complete.

Validation of oscillation criteria for first order ADEs with monotone argument

Consider the ADE in equation (4) with

$$q(t) = -\frac{1}{4e^{2t}} \text{ and } \sigma(t) = 2t+1 \text{ can be written as}$$

$$x'(t) - \frac{1}{4e^{2t}} x(2t+1) = 0 \quad (42)$$

Now applying theorem (4.2) on (4.106) we have

$$\lim_{t \rightarrow 0} \frac{x(2t+1)}{x(t)} \geq \int_M^N -\frac{1}{4e^{2t}} \int_t^{2t+1} -\frac{1}{4e^{2s}} ds dt \quad (43)$$

Considering the left hand side of inequality (43)

$$\lambda(t) = 2t+1$$

$$\lim_{t \rightarrow 0} 2t+1 = 1 \quad (44)$$

Now considering the right hand side of (43)

$$\frac{1}{16} \left[\frac{1}{2e^{2t}} \right]_M^N \left[\frac{1}{2e^{2s}} \right]_t^{2t+1} \quad (45)$$

$$\frac{1}{16} \left[\frac{1}{2e^{2t}} \right]_M^N \left[\frac{1}{2e^{2s}} \right]_t^{\tau(t)} \quad (46)$$

$$\frac{1}{16} \left(\frac{1}{2e^{2N}} - \frac{1}{2e^{2M}} \right) \left[\frac{1}{2e^{2t+1}} - \frac{1}{2e^t} \right] \quad (47)$$

$$\frac{1}{32} \left(\frac{e^{2M} - e^{2N}}{e^{4MN}} \right) \left[\frac{1}{2e^{2t+1}} - \frac{1}{2e^t} \right] \quad (48)$$

Comparing (44) and (48) using inequality (41) we have

$$\frac{1}{32} \left(\frac{e^{2M} - e^{2N}}{e^{4MN}} \right) \left[\frac{1}{2e^{2t+1}} - \frac{1}{2e^t} \right] \leq 1 \quad (49)$$

Case 1

Consider the case when $t = 0$

$$\frac{1}{32} \left(\frac{e^{2M} - e^{2N}}{e^{4MN}} \right) \left[\frac{1}{2e^{2(0)+1}} - \frac{1}{2e^0} \right] \leq 1 \quad (50)$$

$$\frac{1}{32} \left(\frac{e^{2M} - e^{2N}}{e^{4MN}} \right) \left[\frac{1}{2e} - 1 \right] \leq 1 \quad (51)$$

The equalities (51) hold then the ADE in equation (42) is oscillatory

Case 2

Consider the case when $t = \infty$

$$\frac{1}{32} \left(\frac{e^{2M} - e^{2N}}{e^{4MN}} \right) \left[\frac{1}{2e^{2(\infty)+1}} - \frac{1}{2e^\infty} \right] \leq 1 \quad (52)$$

$$\frac{1}{32} \left(\frac{e^{2M} - e^{2N}}{e^{4MN}} \right) \frac{1}{2} \left[\frac{1}{e^\infty} - \frac{1}{e^\infty} \right] \leq 1$$

$$0 \leq 1 \quad (53)$$

The equalities in (53) hold when t is large then the ADE in equation (42) is oscillatory

IV. DISCUSSION

The problem of oscillation of First Order ADE with monotone is addressed in theorem (1) which determines whether equation (4) demonstrates oscillatory behavior, our result is different from the from the main paper reviewed because we considered our argument to be monotone and our equation coefficient is unique, the condition obtained is also unique it differ from the one in existing literature.

Validation involved considering our: $p(t) = \frac{1}{4e^{2t}}$

and $\sigma(t) = 2t+1$ where both the coefficient and the delay are function of time and also the delay is a monotone function. Therefore according to theorem (1) equation (4.102) is oscillatory for both cases that is when $t = 0$ and $t = \infty$

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