

The Application of VaR and TCE in Solving a Portfolio Risk Management Problem in Financial Institutions

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Abstract

Risk management is central to the stability and profitability of financial institutions, particularly in the face of market volatility and uncertain loss distributions. Among the most widely applied risk measures are Value at Risk (VaR) and Tail Conditional Expectation (TCE). VaR provides a probabilistic threshold for potential losses at a given confidence level, making it attractive for regulatory coverage and capital requirement resolve. However, its inability to capture the magnitude of losses beyond this threshold limits its effectiveness in managing extreme risk. TCE addresses this limitation by quantifying the expected loss given that the VaR boundary has been breached, thereby offering a more all-inclusive view of tail risk. This paper investigates the application of VaR and TCE in portfolio risk management when asset returns follow a Pareto distribution, a heavy-tailed model frequently used to capture extreme events in financial markets. Analytical formulations of VaR and TCE under the Pareto framework are derived, and their effectiveness in capturing tail risk is compared through a simulated portfolio problem. The results suggest that financial institutions relying solely on VaR may miscalculate losses, whereas incorporating TCE into risk management frameworks enhances robustness against disastrous market downturns.

Keywords: VaR; TCE; financial institutions; Pareto distribution; risk management

1 Introduction

Financial institutions operate in environments characterized by uncertainty, exposure to extreme losses, and market volatility. Effective risk measurement is therefore essential for safeguarding solvency, meeting regulatory requirements, and maintaining investor confidence. The risk measures imposed by regulators on insurance companies share a common feature: they are all related to the behavior of tails of the probability distribution of a firm's financial losses, which could bring about economic failure. Risk-based capital requirements are a natural outgrowth of a traditional thoughtful regulation aiming at preserving the solvency of financial institutions.

Hence, safety capital should be computed by looking at the extreme risks that can impact financial institutions in general and insurance enterprises in particular. However, regulators differ in their specification of the tail that they recommend. Some regulators enforce the use of quantiles of the distribution (Value at Risk, VaR), while other regulators impose the use of partial moments (Tail Conditional Expectation, TCE). Furthermore, regulators also differ in their choice of time horizon. Finally, they also differ in their choice of confidence level (i.e., probability level for the quantile, or for

the conditional expectation). Solvency II enforces risk-based capital computed using the VaR as a risk measure over a one-year period and with a confidence level of 99.5%. Solvency II is not only the main regulatory law for prudential regulation of insurance enterprises but it has a model for risk-based capital requirements worldwide.

The regulation of risk-based capital for insurers includes a capital standard, and stress-testing of certain acute scenarios. The capital standard is based on the expected shortfall, or equivalently, Tail Conditional Expectation (TCE).

VaR and TCE are the most classic examples of risk measures (Linsmeier and Pearson 2000; Klugman et al. 2012). A risk measure is a map from the random variable representing risk exposure to the set of real numbers. It can be interpreted as the amount of capital required to save from harm against outcomes of a given risk. Artzner et al. (1999) introduced the concept of coherence of risk measurement. A coherent risk measure is defined by the following four properties: subadditivity, monotonicity, positive homogeneity, and translation invariance. The properties of risk measures in the context of insurance are discussed by Wang and Zitikis (2021). Acerbi and Tasche (2002) noted that TCE is a logical risk measure, while VaR is not.

Rostek (2010) issues an interesting alternative to evaluation of risk, a model of preferences, in which, given beliefs about uncertain outcomes, an individual evaluates an action by a quantile of the induced distribution. Fadina et al. (2021) designed a unified axiomatic framework for risk evaluation principles, which quantifies jointly a gross random variable and a set of plausible probabilities. Fuchs et al. (2017) show that a notion of a quantile risk measure is a natural generalization of that of a haunted risk measure and provides another view of the alteration risk measures generated by a distribution function on the unit interval. In this general setting, they prove several results on quantile or spectral risk measures and their domain with special consideration of the expected deficit. They also provide a particular short proof of the subadditivity of the expected deficit risk measure. Denuit et al. (2006) provide a comprehensive review of modeling risk in incomplete markets, with emphasis on insurance risk, expanding on and combining in a comprehensive review of the existing literature on quantitative risk management.

This paper was developed independently from these very many existing papers and develops related comparisons of VaR and TCE quantiles and their application. A major contribution of our paper is the introduction of a risk pointer that extends TCE to take into account higher-order risks. We also provide comparisons of this new pointer with more classic risk indicators. The goal of our paper is to understand how regulators choose such or such risk indicator for solvency computations. Also, to understand the hidden utility function of insurance regulators and to understand how corresponding risk valuation systems can be put in place. The paper makes use of a probabilistic assumption, where we assume claims to be a Pareto distribution. We make these assumptions because they allow us to very suitably derive clear results and they are reliable with what is observed for claims with heavily distributed tails.

2 VaR and TCE

On recalling key preparatory elements on VaR and TCE, we first recall classic definitions, then, examine the correspondence between VaR and TCE and their applications.

2.1 Definitions

The most used risk measure is value at risk (VaR), which is the expected worst loss over a given horizon at a given level of confidence. Linsmeier and Pearson (2000) defined VaR as the loss that is expected to be exceeded with a probability of only $(1 - \alpha)$ percent during the next holding period T . Also, Promislow (2015) expressed VaR as the amount of capital that will be sufficient to ensure that the losses over a given period T will be covered *most of the time* where 'most' is measured by the specified confidence level α . The role of regulators is to choose the value of the confidence level α and of the horizon T .

Suppose that $F_{X_T}(x)$ represents the distribution function of outcomes over a fixed period of time T of an assortment of risks. An adverse outcome is a loss and, in this case, positive values of the random variable X_T are losses. The VaR of the random variable is the α percentile of the distribution of X_T , denoted by:

$$VaR_\alpha(X_T) = F_{X_T}^{-1}(\alpha)$$

Tail conditional expectation (TCE), also referred to as tail value at risk (TVaR), measures the average losses over the defined threshold (typically set as the VaR at a given confidence level α). In other words, TCE is a conditional mean value, given that the loss exceeds the $(1 - \alpha)$ percentile. It is also often called expected shortfall (ES). TCE can be defined as follows:

$$TCE_\alpha(X_T) = E[X_T | X_T \geq VaR_\alpha(X_T)]$$

Furthermore, if the random variable is continuous, we can write:

$$TCE_\alpha(X_T) = \frac{1}{1 - \alpha} \int_\alpha^1 VaR_u(X_T) du$$

2.2 Correspondence between VaR and TCE quantiles

If we compute VaR and TCE using the same quantile, TCE will be always higher than VaR, by construction. However, the quantiles for VaR and TCE are usually chosen to be different. We study the correspondence that exists between these two quantiles. Specifically, we examine how it is possible to find c and q such that $VaR_q = TCE_c$, where $q > c$. We conduct our analysis when risks follow a generalized Pareto distribution with three parameters: scale σ , shape ξ , and location μ . The cumulative distribution function of X_T can be expressed as follows:

$$F(X_T) = 1 - \left(1 + \frac{\xi(x - \mu)}{\sigma}\right)^{-\frac{1}{\xi}} \quad \xi \neq 0 \quad (1)$$

We can equally show that the VaR can be computed as follows:

$$VaR_q(X_T) = \mu + ((1 - q)^{-\xi} - 1) \cdot \frac{\sigma}{\xi} \quad (2)$$

The tail conditional expectation admits the following expression:

To show that $VaR_q(X_T) = TCE_c(X_T)$, we first rewrite $TCE_c(X_T)$ as a function of $VaR_c(X_T)$:

$$TCE_c(X_T) = \frac{1}{1 - \xi} \cdot VaR_c(X_T) + \frac{\sigma - \xi\mu}{1 - \xi}$$

Thus, $TCE_c(X_T) = VaR_q(X_T)$ is equivalent to:

$$\frac{1}{1-\xi} \cdot VaR_c(X_T) + \frac{\sigma - \xi\mu}{1-\xi} - VaR_q(X_T) = 0 \quad (3)$$

Using the above equality, we can relate the quantities c and q as follows.

In the generalized Pareto framework, the quantile of TCE and quantile of VaR obey the following relationship when the two risk indicators are equal:

$$c = 1 - (1 - \xi)^{-\frac{1}{\xi}} \cdot (1 - q) \quad (4)$$

where c depends on $\xi \in (0, 1)$.

We shall note that a higher value of ξ is equivalent to an increased presence of extreme risks in the phenomenon under study. When there is more risk on the market, that is, when $\xi \rightarrow 1$, the VaR quantile has to be a number close to 1, whereas the TCE quantile can take a broad range of values between 0.9 and 1.

This feature implies that in the presence of an extreme risk, there is a large variability in the choice of the TCE quantile, making it a difficult value to choose by regulators. Conversely, in the presence of a lot of risks, the VaR quantile is easy to set, where the regulator only needs to choose a sufficiently high value.

However, this feature is not observed when ξ is small, where a lot of admissible values can be taken by both the VaR and TCE quantiles, and where these two quantiles vary linearly.

Indeed, it is practically not viable to find a pair of reasonable values of the VaR and TCE quantiles that is reliable with both regulations.

3 A high-order TCE pointer

We introduce a generalized TCE indicator, which is a conditional higher-order moment of the probability distribution under study. We compute this indicator when losses follow a generalized Pareto distribution.

3.1 Definition

By using higher-order moments, we construct a higher-order measure of risk as a probability distribution, which is a TCE at order p . We denote this indicator by $TCE^{(p)}$ and we define it as follows:

$$TCE_c^{(p)}(X_T) = \frac{1}{1-c} \int_{VaR_c(X_T)}^{+\infty} x^m dF(x) \quad (5)$$

where $c = F(VaR_c(X_T))$.

Upon changing the variables as follows: $F(x) = s$, $x = F^{-1}(s) = VaR_s(X_T)$ and $ds = dF(x)$,

we readily obtain a third equivalent representation of the extended TCE indicator:

$$TCE_c^{(p)}(X_T) = \frac{1}{1-c} \int_c^1 (VaR_s(X_T))^p ds \quad (6)$$

Let us now assume that losses follow a Pareto distribution, whose probability density function is represented by:

$$f_{X_T}(x) = \frac{\theta^\alpha}{x^{\alpha+1}} \quad (7)$$

where α is the shape parameter and θ the scale parameter.

Applying the definition in Equation (5) to compute $TCE_c^{(m)}$ for a Pareto distribution, we have:

$$TCE_c^{(p)}(X_T) = \frac{1}{1-c} \int_{VaR_c(X_T)}^{+\infty} x^p \frac{\theta^\alpha}{x^{\alpha+1}} dx$$

This expression can be developed in closed form as a function of value at risk, as demonstrated in the theorem below.

Theorem 1. *When losses are Pareto distributed, the extended TCE indicator admits the expression*

$$TCE_c^{(p)}(X_T) = \frac{\alpha}{\alpha - p} (VaR_c(X_T))^p \quad (8)$$

when $\alpha > p$.

Proof. Using the Pareto distribution, we have:

$$\begin{aligned} TCE_c^{(p)}(X_T) &= \frac{1}{1-c} \int_{VaR_c(X_T)}^{+\infty} x^p \frac{\theta^\alpha}{x^{\alpha+1}} dx \\ &= \frac{1}{1-c} \left[-\frac{\alpha\theta^\alpha}{\alpha-p} x^{p-\alpha} \right]_{VaR_c(X_T)}^{+\infty} \end{aligned}$$

The quantity $\lim_{x \rightarrow +\infty} \left(-\frac{\alpha\theta^\alpha}{\alpha-p} x^{p-\alpha} \right)$ converges to zero only if $-\alpha < 0$. Assuming that $\alpha > p$, we obtain

$$TCE_c^{(p)}(X_T) = \frac{1}{1-c} \frac{\alpha\theta^\alpha}{\alpha-p} (VaR_c(X_T))^{p-\alpha}$$

Next, we write:

$$TCE_c^{(p)}(X_T) = \frac{1}{1-c} \frac{\alpha\theta^\alpha}{\alpha-p} (VaR_c(X_T))^{-\alpha} (VaR_c(X_T))^p$$

Now, replacing VaR with its expression:

$$VaR_c(X_T) = \theta(1-c)^{-\frac{1}{\alpha}}$$

to obtain:

$$TCE_c^{(p)}(X_T) = \frac{1}{1-c} \frac{\alpha\theta^\alpha}{\alpha-p} \left(\theta(1-c)^{-\frac{1}{\alpha}} \right)^{-\alpha} (VaR_c(X_T))^p$$

Finally, we have:

$$TCE_c^{(p)}(X_T) = \frac{\alpha}{\alpha-p} (VaR_c(X_T))^p$$

Theorem 2. In the Pareto framework, the extended TCE quantile (c) and the VaR quantile (q) agree with the relation

$$c = 1 - \left(\frac{(\alpha-p)\theta^{1-p}}{\alpha} \right)^{-\frac{\alpha}{p}} (1-q)^{\frac{1}{p}} \quad (10)$$

when the two risk pointers are equal.

Proof. Our aim is to solve Equation (9) when $VaR_c(X_T) = \theta(1 - c)^{-\frac{1}{\alpha}}$, so when X_T follows a Pareto distribution. Substituting VaR with its expression, Equation (9) becomes:

$$\frac{\alpha}{\alpha - m} \cdot (\theta(1 - c)^{-\frac{1}{\alpha}})^p - (\theta(1 - q)^{-\frac{1}{\alpha}}) = 0$$

Next, we have:

$$\frac{\alpha}{\alpha - p} \cdot \theta^p (1 - c)^{-\frac{p}{\alpha}} - \theta(1 - q)^{-\frac{1}{\alpha}} = 0$$

so that

$$c = 1 - \left(\frac{\alpha - p}{\alpha \theta^p} \theta(1 - q)^{-\frac{1}{\alpha}} \right)^{-\frac{\alpha}{p}}$$

which can readily be written as Equation (10).

For consistency with the generalized Pareto approach, we replace the shape parameter α of the classic Pareto distribution with $\xi = \frac{1}{\alpha}$, which can also be interpreted as a shape parameter.

Thus, Equation (10) can be rewritten as follows:

$$c = 1 - \left[(1 - p\xi) \theta^{1-p} \right]^{-\frac{1}{p\xi}} (1 - q)^{\frac{1}{p}} \quad (11)$$

where $0 < \xi < \frac{1}{p}$.

Observe that Equation (11) reduces to Equation (4) when $p = 1$.

4 Pareto distributed losses

To model losses as random variables that follow a classic Pareto distribution, we can use Equation (8) to compute $TCE_{q^{(p)}}^{(p)}(X_T)$ where $\alpha > p$. Thus, we can compare two higher order TCE as follows:

$$\frac{\alpha}{\alpha - p} (VaR_{q^{(p)}}(X_T))^p = \frac{\alpha}{\alpha - n} (VaR_{q^{(n)}}(X_T))^n$$

where

$$VaR_{q^{(p)}}(X_T) = \theta(1 - q^{(p)})^{-\frac{1}{\alpha}}$$

we obtain:

$$\frac{\alpha}{\alpha - p} \theta^p (1 - q^{(p)})^{-\frac{p}{\alpha}} = \frac{\alpha}{\alpha - n} \theta^n (1 - q^{(n)})^{-\frac{n}{\alpha}}$$

which leads us to:

$$q^{(n)} = 1 - \left(\frac{\alpha - n}{\alpha - p} \theta^{p-n} \right)^{-\frac{\alpha}{n}} (1 - q^{(p)})^{-\frac{p}{n}} \quad (12)$$

where $\alpha > p$, n and $\theta > 0$.

We observe that high-order TCE concentrate on different parts of probability tails. Thus, a manager that reduces high-order extreme risks at a given order, say p , is not simultaneously reducing high-order extreme risks at another order, say n .

Specifically, when $n = 1$, we have:

$$\frac{\alpha}{\alpha - 1} (VaR_q(X_T))^1 = \frac{\alpha}{\alpha - m} (VaR_{q^{(p)}}(X_T))^p$$

Equation (12) becomes:

$$q = 1 - \left(\frac{\alpha - p}{(\alpha - 1)\theta^{p-1}} \right)^\alpha (1 - q^{(p)})^{-p}$$

when $0 < \xi < \frac{1}{p}$ and $\theta > 0$.

4.1 Computing VaR and TCE for a Pareto loss model

Example. Suppose an insurance company models its individual claim losses X with a Pareto distribution, having the scale parameter $X_p = N1000000.0$ and shape parameter $\alpha = 3$. Compute (i) the 95% value at risk and (ii) the 95% tail conditional expectation.

Solution. The cumulative distribution function (CDF) of a Pareto (α, x_p) is $F_X(x) = 1 - \left(\frac{x_p}{x} \right)^\alpha$, $x \geq x_p$.

Hence, the quantile function (inverse CDF) is $F_X^{-1}(\alpha) = x_p(1 - \alpha)^{-\frac{1}{\alpha}}$.

(i) Now, to compute VaR at $\alpha = 0.95$:

$$\begin{aligned} VaR_{0.95} &= x_p(1 - 0.95)^{-\frac{1}{3}} = 1,000,000(0.05)^{-\frac{1}{3}} \\ &= N2714000 \end{aligned}$$

$$VaR_{0.95} = N2.714 \text{ million}$$

(ii) To compute $TCE_{0.95}$ for a Pareto distribution ($\alpha > 1$):

$$TCE_\alpha = \frac{\alpha}{\alpha - 1} VaR_\alpha$$

So,

$$\begin{aligned} TCE_{0.95} &= \frac{3}{2} \times 2714000 \\ &= N4071000 \end{aligned}$$

$$TCE_{0.95} = N4.071 \text{ million}$$

It therefore implies that there is a 5% probability that losses exceed N2.714 million and if that happens, the expected loss is N4.071 million.

Hence, this will help the insurer to:

- set its capital reserve (N4.071 million) for 95% coverage;
- understand that ordinary losses are $\leq$ N2.714 million on most days; and
- plan reinsurance or capital allocation above N4 million.

5 Conclusions

We computed VaR, TCE and $TCE^{(p)}$ and compared VaR and TCE with a high-order TCE indicator by adjusting the latter quantity in terms of scale. Also, we introduced a risk indicator that is a high-order TCE risk measure. We compared the quantiles of this indicator to the quantiles of VaR in a simple Pareto framework. We also examined equivalence results between the quantiles of high-order TCEs. By doing so, we aim to illustrate the interplay between implicit choices of risk measures by regulators and the characteristics of probability distribution tails. An example is also given to illustrate our results.

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