

The Role of the Jacobson Radical in Defining the Structure of Prime and Maximal Ideals in Artinian Rings

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Abstract

The Jacobson radical $J(R)$ of a ring R is a fundamental tool in the structural analysis of Artinian rings. This paper examines how $J(R)$ governs the behaviour of prime and maximal ideals in such rings. The basic properties of the radical are established, including the unit characterisation, its behaviour under quotients and its relation to simple modules, together with Nakayama's Lemma and the computation of the radical for matrix rings, direct products and commutative rings. The central finiteness result is then proved: in a left Artinian ring the Jacobson radical is nilpotent. From this it follows that the quotient $R/J(R)$ is semisimple Artinian, and a finite product of fields when R is commutative. These results are used to prove that every prime ideal of a commutative Artinian ring is maximal, and that the Jacobson radical coincides with the nilradical in that setting, with the noncommutative analogue identifying $J(R)$ as the largest nilpotent left ideal of a left Artinian ring. Examples are given throughout, including instances establishing that the Artinian hypothesis is sufficient but not necessary for the identification of the radical with the nilradical, and that it cannot be dropped altogether.

Keywords: Artinian ring; Jacobson radical; maximal ideal; nilpotent ideal; nilradical; prime ideal; semisimple ring

1 Introduction

The Jacobson radical $J(R)$ occupies a central place in the structure theory of rings. Defined as the intersection of the maximal left ideals of R , it measures the failure of a ring to be semisimple, and it provides the bridge between the local data carried by maximal ideals and the global structure of the ring [9], [12]. In rings satisfying the descending chain condition the radical acquires strong finiteness properties, of which nilpotency is the most consequential, and these properties constrain the behaviour of ideals severely [2], [18].

This paper examines the role played by the Jacobson radical in determining the structure of prime and maximal ideals in Artinian rings. Three results organise the exposition. The first is that $J(R)$ is nilpotent whenever R is left Artinian. The second is that every prime ideal of a commutative Artinian ring is maximal. The third is that in a commutative Artinian ring the Jacobson radical coincides with the nilradical. Each of these is classical; the purpose here is to present them as a connected development in which the nilpotency theorem does the principal work, and to record the examples that delimit the hypotheses.

The exposition proceeds as follows. The basic properties of $J(R)$ are collected first, including the unit characterisation, functoriality, the relation to simple modules and Nakayama's Lemma, together with the computation of the radical for matrix rings, direct products and commutative rings. Nilpotency in

the Artinian case is then established and the nilpotency index examined. The quotient $R/J(R)$ and its semisimplicity are treated next. It is then proved that prime ideals are maximal in the commutative Artinian setting, and the radical is identified with the nilradical there, together with the noncommutative analogue. Conclusions are drawn in the final section.

Throughout, R denotes an associative ring with identity $1 \neq 0$, and *ideal* means two-sided ideal unless stated otherwise. For a commutative ring, $\text{Nil}(R)$ denotes the set of nilpotent elements.

2 Preliminaries and basic properties

2.1 The radical and its unit characterisation

Definition 2.1 (Jacobson radical). The *Jacobson radical* $J(R)$ of a ring R is the intersection of all maximal left ideals of R ,

$$J(R) = \bigcap \{M : M \text{ a maximal left ideal of } R\} \quad (1)$$

The same set is obtained by intersecting all maximal right ideals, so the definition is left-right symmetric. For a commutative ring, $J(R)$ is the intersection of all maximal ideals [2, p. 25], [9, p. 195].

Theorem 2.2 (Unit characterisation). Let R be a ring and let $x \in R$. Then $x \in J(R)$ if and only if $1 - rx$ is a unit in R for every $r \in R$. [8, Th. 1.2.3], [9, Th. 4.3].

Proof. ($\Rightarrow$) Suppose $x \in J(R)$ and let $r \in R$. If $1 - rx$ were not a unit, the left ideal $R(1 - rx)$ would be proper and hence contained in a maximal left ideal M , so that $1 - rx \in M$. Since $x \in J(R) \subseteq M$ and M is a left ideal, $rx \in M$, whence $1 = (1 - rx) + rx \in M$, contradicting the maximality of M . Therefore $1 - rx$ is a unit.

($\Leftarrow$) Suppose $1 - rx$ is a unit for every $r \in R$, and let M be a maximal left ideal. If $x \notin M$ then $M + Rx = R$, so $1 = m + rx$ for some $m \in M$ and $r \in R$, giving $1 - rx = m \in M$. As $1 - rx$ is a unit, M would contain a unit and hence equal R , a contradiction. Thus x lies in every maximal left ideal, that is, $x \in J(R)$. $\square$

The characterisation is used repeatedly below, and it is convenient to record it in the form

$$J(R) = \{x \in R : 1 - rx \text{ is a unit for every } r \in R\} \quad (2)$$

Proposition 2.3. For any ring R , $J(R)$ is a two-sided ideal of R . [8, Th. 1.2.1].

Proof. Each maximal left ideal is a left ideal, so the intersection $J(R)$ is closed under subtraction and under left multiplication by elements of R ; that is, $a - b \in J(R)$ and $ra \in J(R)$ for all $a, b \in J(R)$ and $r \in R$. Hence $J(R)$ is a left ideal.

To see that $J(R)$ is also a right ideal, let $a \in J(R)$ and $r \in R$; we must show that $ar \in J(R)$. By Theorem 2.2 it suffices to prove that $1 - s(ar)$ is a unit for every $s \in R$. Fix $s \in R$. Since $a \in J(R)$, applying Theorem 2.2 to the element rs gives that $1 - (rs)a$ is a unit, so there exists $u \in R$ with

$$u(1 - (rs)a) = 1 \quad (3)$$

Now apply the standard identity that $1 - xy$ is invertible if and only if $1 - yx$ is invertible, with $x = sa$ and $y = r$: since $1 - yx = 1 - r(sa) = 1 - (rs)a$ is a unit, so is $1 - xy = 1 - s(ar)$. As s was arbitrary, Theorem 2.2 gives $ar \in J(R)$. Therefore $J(R)$ is a right ideal, and being both a left and a right ideal, it is two-sided. $\square$

Proposition 2.4. $J(R)$ contains every nilpotent left ideal of R . More generally, $J(R)$ contains every left ideal all of whose elements are nilpotent. [8, Lem. 1.2.2], [18, Prop. 2.1].

Proof. Let L be a left ideal every element of which is nilpotent and let $x \in L$. For any $r \in R$ we have $rx \in L$, so $(rx)^n = 0$ for some $n \geq 1$. Then $1 - rx$ is invertible, with

$$(1 - rx)^{-1} = 1 + rx + (rx)^2 + \cdots + (rx)^{n-1} \quad (4)$$

since

$$(1 - rx)(1 + rx + \cdots + (rx)^{n-1}) = 1 - (rx)^n = 1 \quad (5)$$

Thus $1 - rx$ is a unit for every $r \in R$, and Theorem 2.2 gives $x \in J(R)$. As $x \in L$ was arbitrary, $L \subseteq J(R)$. $\square$

Corollary 2.5. $J(R)$ is a proper ideal of R ; in particular $1 \notin J(R)$. [2, Cor. 1.7].

Proposition 2.6. $J(R)$ contains no nonzero idempotent. If $e^2 = e$ and $e \in J(R)$, then $e = 0$. [12, Prop. 4.2].

Proof. Let $e \in J(R)$ with $e^2 = e$. By Theorem 2.2 the element $1 - e$ is a unit. From $e^2 = e$ we obtain $e(1 - e) = 0$, and multiplying on the right by $(1 - e)^{-1}$ gives $e = 0$. $\square$

2.2 Behaviour under homomorphisms and quotients

Theorem 2.7 (Functoriality). Let $\varphi : R \rightarrow S$ be a ring homomorphism. Then $\varphi(J(R)) \subseteq J(S)$. If φ is surjective, then $\varphi(J(R)) = J(S)$. [2, Prop. 1.8], [13, Th. 3.5].

Corollary 2.8. For any ideal $I \subseteq J(R)$ we have $J(R/I) = J(R)/I$. [2, Prop. 1.9].

2.3 The radical and simple modules

Proposition 2.9. Let M be a left R -module. Then $J(R)M$ is contained in every maximal submodule of M . In particular, if M is simple then $J(R)M = 0$. [12, Lem. 4.8].

Corollary 2.10. $J(R) = \{x \in R : xM = 0 \text{ for every simple left } R\text{-module } M\}$. [12, Cor. 4.9].

2.4 Nakayama's lemma

Theorem 2.11 (Nakayama's Lemma). Let R be a ring, let $J = J(R)$, and let M be a finitely generated left R -module. If $JM = M$, then $M = 0$. [2, Prop. 2.6], [13, Th. 2.2].

Proof. Suppose $M \neq 0$ and let $u_1, \dots, u_n$ be a generating set for M of minimal cardinality $n \geq 1$. Since $M = JM$, the element u_n lies in JM , so there is an expression

$$u_n = a_1 u_1 + \cdots + a_n u_n, \quad a_1, \dots, a_n \in J \quad (6)$$

Collecting the last term gives

$$(1 - a_n)u_n = a_1 u_1 + \cdots + a_{n-1} u_{n-1} \quad (7)$$

Because $a_n \in J(R)$, Theorem 2.2 shows that $1 - a_n$ is a unit, so u_n lies in the submodule generated by $u_1, \dots, u_{n-1}$. This contradicts the minimality of n . Hence $M = 0$. $\square$

Corollary 2.12. If M is finitely generated and $N \subseteq M$ is a submodule with $N + JM = M$, then $N = M$. [2, Cor. 2.8].

2.5 Matrix rings, direct products and the commutative case

Theorem 2.13. For any ring R and any positive integer n ,

$$J(M_n(R)) = M_n(J(R)) \quad (8)$$

[8, Th. 2.1.5], [12, Th. 5.7].

Proposition 2.14. Let $\{R_i\}_{i \in I}$ be any family of rings. Then

$$J\left(\prod_{i \in I} R_i\right) = \prod_{i \in I} J(R_i) \quad (9)$$

[2, Prop. 1.10]. No finiteness hypothesis on the index set or on the individual rings is required.

Theorem 2.15 (Commutative case). Let R be a commutative ring. Then

$$J(R) = \{x \in R : 1 - xy \text{ is a unit for all } y \in R\} = \bigcap_{\mathfrak{m} \text{ maximal}} \mathfrak{m} \quad (10)$$

[2, Prop. 1.6].

Proposition 2.16. If R is a commutative local ring with unique maximal ideal $\mathfrak{m}$, then $J(R) = \mathfrak{m}$. [13, Th. 3.2], [19, §2].

Proof. By Definition 2.1 the radical of a commutative ring is the intersection of its maximal ideals. If R is local, that family has the single member $\mathfrak{m}$, so the intersection is $\mathfrak{m}$ itself:

$$J(R) = \bigcap_{\mathfrak{n} \text{ maximal}} \mathfrak{n} = \mathfrak{m} \quad (11)$$

□

Example 2.17 (Local rings). Two standard instances:

- the localisation $\mathbb{Z}_{(p)}$ of the integers at a prime p has $J = p\mathbb{Z}_{(p)}$ [2, Prop. 3.8];
- the formal power series ring $k[[x]]$ over a field k has $J = (x)$.

Example 2.18 (A non-local commutative ring). For $R = \mathbb{Z}$ the maximal ideals are the ideals $p\mathbb{Z}$ with p prime, and

$$J(\mathbb{Z}) = \bigcap_{p \text{ prime}} p\mathbb{Z} = 0 \quad (12)$$

so the Jacobson radical may be trivial even in an infinite ring [2, Ex. 1.4].

Table 1 summarises the properties established in this section.

Table 1. Summary of basic properties of the Jacobson radical.

Property	Statement	Reference
Ideal structure	$J(R)$ is a proper two-sided ideal	Prop. 2.3, Cor. 2.5
Unit criterion	$x \in J(R) \Leftrightarrow 1 - rx$ is a unit for all r	Th. 2.2
Nilpotent ideals	every nil left ideal lies in $J(R)$	Prop. 2.4
Idempotents	$J(R)$ contains no nonzero idempotent	Prop. 2.6
Quotients	$J(R/I) = J(R)/I$ for $I \subseteq J(R)$	Cor. 2.8
Simple modules	$J(R)$ annihilates every simple module	Cor. 2.10

Property	Statement	Reference
Matrix rings	$J(M_n(R)) = M_n(J(R))$	Th. 2.13
Direct products	$J(\prod R_i) = \prod J(R_i)$	Prop. 2.14
Local rings	$J(R) = \mathfrak{m}$ for R commutative local	Prop. 2.16

The entries hold for an arbitrary associative ring with identity unless a commutativity hypothesis is stated.

3 Nilpotency of the radical in Artinian rings

3.1 The descending chain condition

Definition 3.1 (Artinian ring). A ring R is *left Artinian* if every descending chain of left ideals

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots \quad (13)$$

stabilises, that is, $I_n = I_{n+1} = \cdots$ for some n . Right Artinian and two-sided Artinian rings are defined analogously; for commutative rings the three notions coincide [2, p. 89], [13, p. 115].

Theorem 3.2 (Nilpotency of $J(R)$). *If R is a left Artinian ring, then $J(R)$ is nilpotent: there exists a positive integer n with $J(R)^n = 0$.* [2, Prop. 8.4], [10, §2], [18, Th. 3.1].

Proof. Consider the descending chain of two-sided ideals

$$J(R) \supseteq J(R)^2 \supseteq J(R)^3 \supseteq \cdots \quad (14)$$

Since R is left Artinian the chain stabilises, so there is $n \geq 1$ with $J(R)^n = J(R)^{n+1}$. Put $I = J(R)^n$, so that $I = J(R)I$. By the Hopkins–Levitzki theorem (Theorem 4.2 below) a left Artinian ring is left Noetherian, so every left ideal of R is finitely generated; in particular I is a finitely generated left R -module satisfying $J(R)I = I$. Nakayama’s Lemma (Theorem 2.11) gives $I = 0$, that is,

$$J(R)^n = 0 \quad (15)$$

□

The appeal to Theorem 4.2 is not circular: the Hopkins–Levitzki theorem is proved from the descending chain condition alone and does not use the present theorem.

Corollary 3.3. *In a left Artinian ring every element of $J(R)$ is nilpotent.* [12, Cor. 4.13], [18, Cor. 3.2].

Corollary 3.4. *A left Artinian ring R is semisimple if and only if $J(R) = 0$, in which case R is a finite direct product of matrix rings over division rings.* [12, Th. 4.14], [16, §3].

3.2 The index of nilpotency

Definition 3.5. The least positive integer n with $J(R)^n = 0$ is the *nilpotency index* of $J(R)$.

Proposition 3.6. *If R is left Artinian and $J(R) \neq 0$, then the nilpotency index n satisfies $n \geq 2$, and $J(R)^{n-1}$ is a nonzero nilpotent ideal.*

Proof. If n were 1 then $J(R) = J(R)^1 = 0$, contrary to hypothesis; hence $n \geq 2$. By minimality of n , $J(R)^{n-1} \neq 0$, and it is nilpotent because its square is contained in $J(R)^n = 0$ whenever $n \geq 2$. □

Proposition 3.7. *Let R be a commutative Artinian ring. Then the nilpotency index of $J(R)$ is the maximum of the nilpotency indices of the local factors in the decomposition of R as a finite product of local Artinian rings. [13, Th. 4.5].*

3.3 Examples

Example 3.8 (Upper triangular matrices). Let k be a field and let

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b, c \in k \right\} \quad (16)$$

be the ring of upper triangular 2×2 matrices over k . Then

$$J(R) = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} : b \in k \right\} \quad (17)$$

and a direct computation gives $J(R)^2 = 0$, so the nilpotency index is 2 [12, Ex. 4.1].

Example 3.9 (Truncated polynomial ring). For $R = k[x]/(x^n)$ we have $J(R) = (x)$, and

$$J(R)^n = (x^n) = 0, \quad J(R)^{n-1} = (x^{n-1}) \neq 0 \quad (18)$$

so the nilpotency index is n [2, Ex. 8.6].

Example 3.10 (A product). Let $R = k[x]/(x^2) \times \mathbb{C}[y]/(y^3)$. Then $J(R) = (x) \times (y)$ and the nilpotency index is $\max(2, 3) = 3$, in accordance with Proposition 3.7.

Example 3.11 (Finite rings). For $R = \mathbb{Z}/p^k\mathbb{Z}$ with p prime we have $J(R) = p\mathbb{Z}/p^k\mathbb{Z}$, and

$$J(R)^k = (p^k) = 0, \quad J(R)^{k-1} = (p^{k-1}) \neq 0 \quad (19)$$

so the nilpotency index is k [2, Ex. 8.2].

3.4 A preliminary relation with the nilradical

Proposition 3.12. *In any commutative ring R ,*

$$\text{Nil}(R) \subseteq J(R) \quad (20)$$

[2, Prop. 1.8].

Proof. Let x be nilpotent, say $x^n = 0$, and let $r \in R$. Then

$$(1 - rx)(1 + rx + r^2x^2 + \cdots + r^{n-1}x^{n-1}) = 1 - r^n x^n = 1 \quad (21)$$

so $1 - rx$ is a unit, and Theorem 2.2 gives $x \in J(R)$. $\square$

The reverse inclusion requires the descending chain condition and is proved in the section on the radical and the nilradical.

4 Semisimplicity and the quotient $R/J(R)$

Definition 4.1 (Semisimple ring). A ring R is *semisimple* if it is a finite direct sum of simple left ideals; equivalently, if every left R -module is projective. For Artinian rings this is equivalent to $J(R) = 0$ [12, p. 45].

Theorem 4.2 (Hopkins–Levitzki). *A left Artinian ring is left Noetherian. Consequently every left ideal of a left Artinian ring is finitely generated. [12, Th. 4.15], [16, Lem. 2.3].*

Theorem 4.3. *If R is left Artinian, then $R/J(R)$ is semisimple Artinian. [12, Th. 4.14], [16, Th. 3.1].*

Proof. Since R is left Artinian, every descending chain of left ideals of R terminates, and the same therefore holds for the quotient $R/J(R)$, which is consequently left Artinian. By Corollary 2.8 applied with $I = J(R)$,

$$J(R/J(R)) = J(R)/J(R) = 0 \quad (22)$$

A ring is semisimple Artinian precisely when it is left Artinian with vanishing Jacobson radical (Definition 4.1 and Corollary 3.4). Both conditions hold for $R/J(R)$, which is therefore semisimple Artinian. $\square$

Corollary 4.4 (Artin–Wedderburn). *If R is left Artinian, then*

$$R/J(R) \cong M_{n_1}(D_1) \times M_{n_2}(D_2) \times \cdots \times M_{n_k}(D_k) \quad (23)$$

where each D_i is a division ring and $n_i \geq 1$ [12, Th. 3.5], [16, Cor. 3.2].

Corollary 4.5. *If R is commutative Artinian, then*

$$R/J(R) \cong F_1 \times F_2 \times \cdots \times F_k \quad (24)$$

where each F_i is a field [13, Th. 4.3].

4.1 Examples

Example 4.6. For $R = \mathbb{Z}/12\mathbb{Z}$ we have $J(R) = 6\mathbb{Z}/12\mathbb{Z} = \{0, 6\}$, and

$$R/J(R) \cong \mathbb{Z}/6\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \quad (25)$$

a product of fields, hence semisimple [2, Ex. 8.3].

Example 4.7. For the ring R of upper triangular 2×2 matrices over a field k of Example 3.8, $R/J(R) \cong k \times k$, which is semisimple [12, Ex. 4.2].

Example 4.8. For $R = \mathbb{Z}/p^k\mathbb{Z}$ with p prime, $R/J(R) \cong \mathbb{Z}/p\mathbb{Z}$, a field.

Example 4.9. For $R = k[x, y]/(x^2, xy, y^2)$ we have $J(R) = (x, y)$, nilpotent of index 2, and $R/J(R) \cong k$ [1, §2].

5 Prime ideals are maximal in commutative Artinian rings

Theorem 5.1. *Let R be a commutative Artinian ring. Then every prime ideal of R is maximal. [2, Prop. 8.1], [15, Th. 2.1].*

Proof. Let P be a prime ideal of R . By Theorem 3.2 there is $n \geq 1$ with $J(R)^n = 0$, so $J(R)^n \subseteq P$. Since P is prime and $J(R)^n = J(R) \cdots J(R)$ is a product of n copies of $J(R)$, primality gives $J(R) \subseteq P$.

Write $\bar{R} = R/J(R)$ and $\bar{P} = P/J(R)$. The ideal $\bar{P}$ is prime in $\bar{R}$, because the correspondence between ideals of R containing $J(R)$ and ideals of $\bar{R}$ preserves primality. By Corollary 4.5,

$$\bar{R} \cong F_1 \times F_2 \times \cdots \times F_t \quad (26)$$

a finite product of fields. In such a product every prime ideal has the form

$$F_1 \times \cdots \times F_{i-1} \times \{0\} \times F_{i+1} \times \cdots \times F_t \quad (27)$$

for some i , and such an ideal is maximal because the corresponding quotient is the field F_i . Hence $\bar{P}$ is maximal in $\bar{R}$. Applying the ideal correspondence in the reverse direction, its preimage P is maximal in R . $\square$

5.1 Consequences

Corollary 5.2. *In a commutative Artinian ring the set of prime ideals coincides with the set of maximal ideals.* [2, Prop. 8.1].

Corollary 5.3. *A commutative Artinian ring has only finitely many prime ideals.* [2, Prop. 8.3], [13, Th. 4.4].

Corollary 5.4. *In a commutative Artinian ring the Jacobson radical equals the intersection of all prime ideals, that is, the nilradical.*

Corollary 5.4 follows from Corollary 5.2 together with the general fact that the nilradical of a commutative ring is the intersection of its prime ideals; it is proved independently, and by a different route, as Theorem 6.1 below.

5.2 Examples

Example 5.5. For $R = \mathbb{Z}/8\mathbb{Z}$ the only prime ideal is $(2) = \{0, 2, 4, 6\}$, which is maximal, and $J(R) = (2)$ [2, Ex. 8.2].

Example 5.6. For $R = k[x]/(x^3)$ the only prime ideal is (x) , which is maximal since $R/(x) \cong k$.

Example 5.7. For $R = k[x]/(x^2) \times k[y]/(y^2)$ the prime ideals are $(x) \times k[y]/(y^2)$ and $k[x]/(x^2) \times (y)$. Both quotients are isomorphic to k , so both ideals are maximal.

6 The radical equals the nilradical in Artinian rings

Theorem 6.1. *For any commutative Artinian ring R , $J(R) = \text{Nil}(R)$.* [18, Th. 4.1].

Proof. The inclusion $\text{Nil}(R) \subseteq J(R)$ holds in any commutative ring by Proposition 3.12, and does not use the chain condition. For the reverse inclusion, let $x \in J(R)$ and consider the descending chain of principal ideals

$$(x) \supseteq (x^2) \supseteq (x^3) \supseteq \dots \quad (28)$$

Since R is Artinian the chain stabilises, so there is $n \geq 1$ with $(x^n) = (x^{n+1})$. Hence $x^n = rx^{n+1}$ for some $r \in R$, and rearranging gives

$$x^n - rx^{n+1} = x^n(1 - rx) = 0 \quad (29)$$

Because $x \in J(R)$, Theorem 2.2 shows that $1 - rx$ is a unit. Multiplying by its inverse yields $x^n = 0$, so x is nilpotent and $x \in \text{Nil}(R)$. $\square$

Corollary 6.2. *In a commutative Artinian ring, $J(R)$ consists precisely of the nilpotent elements.*

Corollary 6.3. *A commutative Artinian ring is reduced if and only if it is semisimple, if and only if it is a finite product of fields.*

6.1 The noncommutative analogue

Theorem 6.4. *In a left Artinian ring R , $J(R)$ is the largest nilpotent left ideal: it is nilpotent and contains every nilpotent left ideal.* [8, Th. 1.2.5], [12, Th. 4.12], [7, §2].

Proof. By Theorem 3.2, $J(R)$ is nilpotent. If L is any nilpotent left ideal then $L \subseteq J(R)$ by Proposition 2.4. Hence $J(R)$ is the unique largest nilpotent left ideal. $\square$

Remark 6.5. In a general noncommutative ring the lower nilradical, defined as the sum of all nilpotent ideals, is contained in $J(R)$ but need not equal it. Equality holds in the left Artinian case by Theorem 6.4.

6.2 Examples and sharpness of the hypotheses

Example 6.6. For $R = \mathbb{Z}/12\mathbb{Z}$ the nilpotent elements are 0 and 6, since $6^2 = 36 \equiv 0 \pmod{12}$. Thus $\text{Nil}(R) = \{0, 6\} = (6) = J(R)$, as Theorem 6.1 requires [2, Ex. 8.7].

Example 6.7. For the upper triangular matrix ring of Example 3.8,

$$\text{Nil}(R) = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} : b \in k \right\} = J(R) \quad (30)$$

illustrating Theorem 6.4 in a noncommutative setting [12, Ex. 4.1], [17, §3].

Example 6.8 (Artinianity is sufficient but not necessary). For $R = \mathbb{Z}$ we have $\text{Nil}(\mathbb{Z}) = 0 = J(\mathbb{Z})$, so the conclusion of Theorem 6.1 holds although $\mathbb{Z}$ is not Artinian.

Example 6.9 (Without Artinianity the conclusion may fail). For the localisation $R = \mathbb{Z}_{(p)}$ we have $J(R) = p\mathbb{Z}_{(p)} \neq 0$ by Example 2.17, whereas $\text{Nil}(R) = 0$ because R is an integral domain. Here R is not Artinian, and the two ideals differ.

Examples 6.8 and 6.9 together show that the Artinian hypothesis in Theorem 6.1 is sufficient but not necessary, and that it cannot simply be dropped.

7 Conclusion

The development above isolates the Jacobson radical as the organising invariant of the Artinian case. Four results carry the argument. In a left Artinian ring the radical is nilpotent (Theorem 3.2). The quotient $R/J(R)$ is semisimple Artinian (Theorem 4.3), and is a finite product of fields when R is commutative (Corollary 4.5). Every prime ideal of a commutative Artinian ring is maximal (Theorem 5.1). And in that setting the radical coincides with the nilradical (Theorem 6.1).

Taken together these results characterise commutative Artinian rings as the finite products of local Artinian rings whose maximal ideals are nilpotent [2, Th. 8.7]. The radical carries the whole of the nilpotent structure, while the quotient by it carries the semisimple, reduced residue; the two parts are complementary, and the nilpotency theorem is what forces the separation. The noncommutative analogue of Theorem 6.4 shows that the same division persists beyond the commutative case, with the radical appearing as the largest nilpotent left ideal, although the identification with a nilradical defined through nilpotent ideals then requires the chain condition.

Two directions suggest themselves. The first is the behaviour of the radical under standard constructions over Artinian base rings, where its nilpotency may or may not be inherited. The second is the quantitative question raised by Proposition 3.7: the nilpotency index is determined by the local

factors, and it would be of interest to bound it in terms of more accessible invariants of the ring, such as its length as a module over itself.

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