

MATHEMATICS

The Eight Vertices of a Cube: A Connection Between Coordinate Geometry, Vector Algebra and the Algebraic Structure Z_2^3

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Published October 2026

Abstract

This paper investigates the correspondence between the eight vertices of a unit cube, their coordinate representations in three-dimensional Euclidean space, their associated vectors, and the finite algebraic structure $Z_2 \times Z_2 \times Z_2$. Each vertex has coordinates whose entries belong to $\{0,1\}$. The standard basis vectors i , j , and k represent the coordinate axes, while the eight binary triples form a vector space of dimension three over the field Z_2 . The paper distinguishes ordinary vector addition over the real numbers from componentwise addition modulo 2, establishes basic algebraic properties, and discusses applications to digital electronics, coding theory, graph theory, and finite algebra.

Keywords: coordinate geometry; cube vertices; vector algebra; binary vectors; Z_2^3 ; finite fields; Boolean algebra; digital electronics; coding theory

1 Introduction

Coordinate geometry represents points in three-dimensional space by ordered triples (x,y,z) . In ordinary Euclidean vector algebra, the standard basis vectors are $i = (1,0,0)$, $j = (0,1,0)$, and $k = (0,0,1)$. Thus the position vector of a point (x,y,z) is $r = x i + y j + z k$. For example, $(1,0,0)$ corresponds to i , $(1,1,0)$ corresponds to $i + j$, and $(1,1,1)$ corresponds to $i + j + k$.

If each coordinate is restricted to 0 or 1, there are exactly $2^3 = 8$ triples. These are the vertices of the unit cube $[0,1]^3$. The same set of triples can be viewed algebraically as the vector space Z_2^3 , provided that addition and scalar multiplication are performed modulo 2. This dual viewpoint links a familiar geometric object with a finite algebraic system.

2 Objectives

- To list and interpret the eight vertices of the unit cube.
- To express each vertex as a vector using i , j , and k .
- To define the binary vector space Z_2^3 and distinguish it from the real vector space R^3 .
- To prove elementary algebraic properties of the binary triples.
- To explain applications in digital systems, coding theory, and cube graphs.

3 The eight vertices of the unit cube

The unit cube is the set of points (x,y,z) for which $0 \leq x,y,z \leq 1$. Its vertices occur when every coordinate is either 0 or 1. The complete list is given in Table 1.

Table 1. The eight vertices of the unit cube and their position vectors.

Binary triple / coordinate point	Euclidean position vector	Geometric description
(0,0,0)	0	Origin
(1,0,0)	i	Unit step along x-axis
(0,1,0)	j	Unit step along y-axis
(0,0,1)	k	Unit step along z-axis
(1,1,0)	i + j	Vertex in the xy-plane
(1,0,1)	i + k	Vertex in the xz-plane
(0,1,1)	j + k	Vertex in the yz-plane
(1,1,1)	i + j + k	Opposite vertex to the origin

4 The algebraic structure Z_2^3

Let $Z_2 = \{0,1\}$ be the field with two elements. Its addition and multiplication are taken modulo 2. In particular, $1 + 1 = 0$, $1 + 0 = 1$, and $1 \cdot 1 = 1$. Define $V = Z_2 \times Z_2 \times Z_2$. An element of V is a triple (a,b,c) , where $a,b,c \in Z_2$.

For $u = (a,b,c)$ and $v = (d,e,f)$, define componentwise addition by $u + v = (a+d, b+e, c+f)$, with each coordinate calculated modulo 2. Scalar multiplication by $\lambda \in Z_2$ is $\lambda u = (\lambda a, \lambda b, \lambda c)$. Under these operations V is a vector space over Z_2 .

The set V has eight elements, but its addition is not ordinary Euclidean vector addition. For example, in R^3 , $i + i = (2,0,0)$; in Z_2^3 , $(1,0,0) + (1,0,0) = (0,0,0)$. The coordinate labels are the same, but the arithmetic is different.

5 Addition table for the eight binary triples

Table 2 gives $u + v$ in Z_2^3 . Each entry is the componentwise sum modulo 2.

Table 2. Addition table for the eight binary triples in Z_2^3 .

+	000	100	010	001	110	101	011	111
000	000	100	010	001	110	101	011	111
100	100	000	110	101	010	001	111	011
010	010	110	000	011	100	111	001	101
001	001	101	011	000	111	100	010	110
110	110	010	100	111	000	011	101	001
101	101	001	111	100	011	000	110	010
011	011	111	001	010	101	110	000	100

+	000	100	010	001	110	101	011	111
111	111	011	101	110	001	010	100	000

6 Fundamental properties

Proposition 1. *The set Z_2^3 is an abelian group under binary addition.*

Proof. Addition is defined coordinatewise using addition in Z_2 . Since addition in Z_2 is associative and commutative, the same properties hold in every coordinate of a triple. The zero vector $(0,0,0)$ is the identity. Every vector is its own additive inverse because $(a,b,c)+(a,b,c) = (0,0,0)$, as each coordinate satisfies $a+a=0$ in Z_2 . Therefore Z_2^3 is an abelian group. ■

Proposition 2. *Z_2^3 is a vector space of dimension three over Z_2 .*

Proof. The vector-space axioms follow coordinatewise from the field axioms of Z_2 . The vectors $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$ span every triple: $(a,b,c)=a(1,0,0)+b(0,1,0)+c(0,0,1)$. They are linearly independent, because $a(1,0,0)+b(0,1,0)+c(0,0,1)=(0,0,0)$ implies $a=b=c=0$. Hence they form a basis and $\dim(Z_2^3)=3$. ■

Proposition 3. *The eight binary triples correspond bijectively to the eight vertices of the unit cube.*

Proof. Map each triple $(a,b,c) \in Z_2^3$ to the point with the same coordinates in $[0,1]^3$. Each coordinate is either 0 or 1, so the image is a cube vertex. Distinct triples give distinct points, and every cube vertex has such a triple. Thus the correspondence is one-to-one and onto. This is a bijection of sets; it does not claim that Euclidean geometry and binary vector addition are the same operation. ■

7 Worked examples

7.1 Example 1: the vertex $(1,1,1)$

In Euclidean vector algebra, its position vector is $r = i + j + k$. In binary vector notation it is $(1,1,1)$. Adding it to itself in Z_2^3 gives $(1,1,1)+(1,1,1)=(0,0,0)$. By contrast, the Euclidean vector added to itself is $(2,2,2)$.

7.2 Example 2: addition of two vertices in Z_2^3

Let $u=(1,1,0)$ and $v=(1,0,1)$. Then $u+v=(1+1,1+0,0+1)=(0,1,1)$, with all coordinates reduced modulo 2. Thus $110 + 101 = 011$.

7.3 Example 3: Hamming distance and cube edges

Two binary triples are joined by an edge of the cube exactly when they differ in one coordinate. For example, 000 and 100 differ in one coordinate, so they are adjacent. The Hamming distance between two binary triples is the number of coordinates in which they differ.

8 Geometric and algebraic interpretations

The cube has eight vertices, twelve edges, and six square faces. Its vertices can be labelled by binary triples. An edge corresponds to changing exactly one bit, which is equivalent to adding one of the basis vectors 100, 010, or 001 in Z_2^3 . This gives a direct link between cube geometry and the Cayley graph of the additive group Z_2^3 .

However, the binary operation is an algebraic operation on labels, not ordinary displacement in Euclidean space. For example, adding 100 modulo 2 to 100 returns 000, whereas moving one unit in the positive x-direction twice in Euclidean space reaches (2,0,0), outside the unit cube.

9 Applications

9.1 Digital electronics

A triple such as (1,0,1) can encode three binary signals. Bitwise addition modulo 2 is the XOR operation: $1 \text{ XOR } 1 = 0$ and $1 \text{ XOR } 0 = 1$. This is useful in parity computation and digital logic.

9.2 Coding theory

Binary triples are binary words of length three. The Hamming distance measures the number of bit positions at which two words differ. This distance is fundamental in detecting and correcting transmission errors.

9.3 Graph theory

The cube graph Q_3 has one vertex for each binary triple, with two vertices adjacent when they differ in exactly one bit. Every vertex has degree three, and the graph has eight vertices and twelve edges.

9.4 Finite algebra

The additive group of Z_2^3 is an elementary abelian group of order eight. It is not the cyclic group of order eight: every element satisfies $u+u=0$. The same set of triples also forms a three-dimensional vector space over the field Z_2 .

9.5 Coordinate geometry and linear algebra

The correspondence helps students move between ordered coordinates, basis-vector notation, and abstract vector spaces. It also demonstrates that a single set can support different mathematical interpretations depending on the operations defined on it.

10 Limitations and clarifications

The notation Z_2^3 means $Z_2 \times Z_2 \times Z_2$, with each coordinate calculated modulo 2. It should not be confused with the eight vertices treated as points in R^3 with ordinary real-number arithmetic. The map from binary triples to cube vertices is a bijection, but it does not preserve Euclidean vector addition as a binary operation.

Also, the cube vertices alone are not a real vector subspace of R^3 , because, for example, $(1,0,0)+(1,0,0)=(2,0,0)$, which is not a vertex. The set of cube vertices is closed under addition modulo 2, not under ordinary real vector addition.

11 Digital algebra and three-bit systems

Digital algebra studies algebraic systems built from discrete symbols, especially binary values 0 and 1, together with precisely defined operations. The field $Z_2 = \{0,1\}$ has two elements. When three binary coordinates are used, Z_2^3 models three-bit words or three digital signals. The eight cube vertices can therefore be interpreted as the eight possible states of a three-bit digital system.

11.1 Binary operations and logic gates

In digital algebra, addition in Z_2 is the exclusive-OR (XOR) operation: $0+0=0$, $0+1=1$, $1+0=1$, and $1+1=0$. Multiplication in Z_2 agrees with the AND operation: $0\cdot0=0$, $0\cdot1=0$, $1\cdot0=0$, and $1\cdot1=1$. Boolean OR is different from addition modulo 2 because $1 \text{ OR } 1 = 1$, whereas $1+1$ in Z_2 equals 0. Thus Boolean OR must not be confused with field addition. The three operations are compared in Table 3.

Table 3. Addition modulo 2 (XOR), multiplication (AND) and Boolean OR on Z_2 .

a	b	a+b in Z_2 (XOR)	a·b (AND)	a OR b
0	0	0	0	0
0	1	1	0	1
1	0	1	0	1
1	1	0	1	1

11.2 The cube as the state space of a digital device

A triple (a,b,c) can represent three switches, three bits in a register, or three binary sensors. For instance, 101 means that the first and third bits are 1 and the second bit is 0. The eight triples 000, 001, 010, 011, 100, 101, 110, and 111 give the complete state space. Changing exactly one bit moves between adjacent vertices of the cube graph.

11.3 Worked digital-algebra example

Let $u = (1,1,0)$ and $v = (1,0,1)$. Bitwise XOR gives $u \oplus v = (1\oplus1, 1\oplus0, 0\oplus1) = (0,1,1)$, or 011. If the same words are combined bitwise using AND, the result is $(1\cdot1, 1\cdot0, 0\cdot1) = (1,0,0)$, or 100. These are different operations on the same set of eight binary triples.

Proposition 4. *The XOR operation forms an abelian group on the three-bit states.*

Proof. XOR is componentwise addition in Z_2 . It is associative and commutative because addition in Z_2 has these properties. The identity is 000, and every word is its own inverse because $u\oplus u=000$. Hence the three-bit words form an abelian group of order eight under XOR. This is the additive group of the vector space Z_2^3 . ■

11.4 Applications

This model is useful in digital circuit design, parity-bit calculations, error detection, finite-state machines, binary encodings, and cube networks. A cube edge represents a one-bit change; the Hamming distance between two states counts the minimum number of bit changes needed to move between them. The algebraic model is exact for binary operations, while the geometric cube provides a visual representation of state transitions.

12 Two-dimensional example: the unit square

For a two-dimensional unit square, the four vertices are $(0,0)$, $(1,0)$, $(0,1)$, and $(1,1)$. In R^2 , with basis vectors $i=(1,0)$ and $j=(0,1)$, these correspond to 0, i , j , and $i+j$. The set $\{0,1\}^2$ is also Z_2^2 when addition is modulo 2. For example, $10+11=01$ in Z_2^2 , while ordinary real-vector addition gives $(1,0)+(1,1)=(2,1)$. The square has four vertices and four edges.

13 Four-dimensional example: the tesseract

The pattern extends naturally to four dimensions. The set $\{0,1\}^4$ has $2^4 = 16$ binary vectors, corresponding to the 16 vertices of a four-dimensional unit hypercube (tesseract). In ordinary $\mathbb{R}^4$, the standard basis vectors are $e_1=(1,0,0,0)$, $e_2=(0,1,0,0)$, $e_3=(0,0,1,0)$, and $e_4=(0,0,0,1)$. A point (a,b,c,d) has position vector $ae_1+be_2+ce_3+de_4$. For example, $(1,0,1,1)$ corresponds to $e_1+e_3+e_4$, and $(1,1,1,1)$ corresponds to $e_1+e_2+e_3+e_4$. Interpreted in $\mathbb{Z}_2^4$, these same tuples use modulo-2 arithmetic.

The four-dimensional analogue of the cube is called a tesseract or 4-cube. It has 16 vertices, 32 edges, 24 square faces, and 8 cubic cells. These counts can be obtained from the formula for the number of k -dimensional faces of an n -dimensional cube: 2^{n-k} times the binomial coefficient $C(n, k)$. In particular, for $n = 4$, the number of vertices is $2^4=16$ and the number of edges is $2^3 \cdot C(4,1)=32$.

13.1 Geometry and algebra in four dimensions

Under componentwise addition modulo 2, the 16 four-bit words form the vector space $\mathbb{Z}_2^4$. For example, $1011 + 1101 = 0110$ because each coordinate is added modulo 2. Every word added to itself gives 0000. Two tesseract vertices are connected by an edge when their labels differ in exactly one bit. Each vertex has four incident edges, one for each coordinate direction.

14 The general n-dimensional pattern

These examples establish the general rule: in dimension n , the vertices of the unit n -cube are exactly the 2^n binary coordinate tuples in $\{0,1\}^n$. They can be viewed geometrically as points in $\mathbb{R}^n$ or algebraically as vectors in $\mathbb{Z}_2^n$, but the operations must be specified. Over $\mathbb{R}$, vector addition is ordinary addition; over $\mathbb{Z}_2$, coordinates are added modulo 2. The binary vector space has dimension n and 2^n elements.

For $n=2$, the square has four vertices; for $n=3$, the cube has eight vertices; and for $n=4$, the tesseract has sixteen vertices. In each case, changing exactly one coordinate from 0 to 1 or from 1 to 0 moves along an edge. The corresponding graph is the n -dimensional hypercube graph Q_n .

14.1 Comparison of dimensions

The unit square, cube, tesseract and general hypercube are compared in Table 4, and examples of points with their position vectors and binary labels are given in Table 5.

Table 4. Comparison of hypercubes in different dimensions.

Dimension	Geometric object	Number of vertices	Binary vector space
2	Unit square	$2^2 = 4$	$\mathbb{Z}_2^2$
3	Unit cube	$2^3 = 8$	$\mathbb{Z}_2^3$
4	Tesseract (4-cube)	$2^4 = 16$	$\mathbb{Z}_2^4$
n	n -dimensional hypercube	2^n	$\mathbb{Z}_2^n$

Table 5. Example points with their Euclidean position vectors and binary labels.

Dimension	Example point	Euclidean position vector	Binary label
2	(1,1)	$i + j$	11

Dimension	Example point	Euclidean position vector	Binary label
3	(1,1,1)	$i + j + k$	111
4	(1,0,1,1)	$e_1 + e_3 + e_4$	1011
4	(1,1,1,1)	$e_1 + e_2 + e_3 + e_4$	1111

15 Conclusion

The eight vertices of a unit cube provide a clear and instructive connection between coordinate geometry, vector algebra, finite algebra, and digital algebra. In Euclidean space, (1,0,0), (0,1,0), and (0,0,1) correspond to i , j , and k , while (1,1,1) corresponds to $i+j+k$. Interpreted over Z_2 , the same coordinate triples form the vector space Z_2^3 with eight elements. This structure explains binary addition, XOR operations, cube-graph adjacency, and coding-theory concepts; it also models the eight states of a three-bit digital system. The key lesson is that the same coordinate labels can describe different structures when the underlying operations change.

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