

Green's Function and a Lyapunov-Type Inequality for a Second-Order Boundary Value Problem with Robin Conditions

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Abstract

We study the second-order differential equation $x''(t) + q(t)x(t) = 0$, $a < t < b$, subject to separated Robin boundary conditions. Under explicit assumptions on the boundary coefficients, we construct the Green's function for the auxiliary operator $-d^2/dt^2$, establish its positivity and symmetry, and derive an integral representation for solutions. This representation yields a Lyapunov-type necessary condition, a weighted eigenvalue estimate, and an L^p -version. Dirichlet and mixed boundary conditions are recovered as special cases. The results provide a self-contained baseline for further work on optimal constants and more general differential operators.

Keywords: Green's function; Lyapunov inequality; Robin boundary conditions; mixed boundary conditions; second-order differential equation; eigenvalue estimate

MSC 2020: 34B05, 34B27, 34C10, 34L15.

1 Introduction

Lyapunov-type inequalities provide necessary conditions for the existence of nontrivial solutions of differential equations subject to prescribed boundary conditions. They have applications in oscillation theory, disconjugacy, and eigenvalue estimation. The classical Dirichlet inequality states that if

$$x''(t) + q(t)x(t) = 0, \quad a < t < b, \quad x(a) = x(b) = 0$$

has a nontrivial solution, then

$$\int_a^b |q(t)| dt > \frac{4}{b-a}$$

Green's functions offer a systematic approach to deriving related inequalities because they represent solutions as integral operators whose norms can be estimated.

The purpose of this article is to provide a rigorous Green's-function derivation for separated Robin conditions. We emphasize that the basic norm estimate proved here is a baseline result; its presence alone does not establish novelty relative to the existing literature.

Related literature includes work on Robin Lyapunov inequalities for elliptic and quasilinear problems [1, 2], a recent paper on generalized fractional Hill equations [3], and work on Lyapunov inequalities for general second-order elliptic equations [4].

2 Problem and assumptions

Let $a < b$ and $L = b - a$. We consider

$$\begin{cases} x''(t) + q(t)x(t) = 0, & a < t < b, \\ \alpha x(a) - \beta x'(a) = 0, \\ \gamma x(b) + \delta x'(b) = 0, \end{cases} \quad (1)$$

where $q \in C[a, b]$ and $\alpha, \beta, \gamma, \delta$ are real constants. Throughout, assume:

(A1) $q \in C[a, b]$;

(A2) $\alpha, \beta, \gamma, \delta \geq 0$;

(A3)

$$D := \alpha\gamma L + \beta\gamma + \alpha\delta > 0. \quad (2)$$

The last condition is a nonresonance condition for the auxiliary problem $-u'' = f$ with the same boundary conditions. The nonnegative coefficient assumption is a sufficient condition for positivity of the Green's function in the formula below; it is not necessary for all Robin problems.

A nontrivial solution means a function $x \in C^2[a, b]$, not identically zero, satisfying (1) and the boundary conditions.

3 Green's function

Consider the auxiliary boundary value problem

$$\begin{cases} -u''(t) = f(t), & a < t < b, \\ \alpha u(a) - \beta u'(a) = 0, \\ \gamma u(b) + \delta u'(b) = 0. \end{cases} \quad (3)$$

Define

$$\phi(t) = \beta + \alpha(t - a), \quad \psi(t) = \delta + \gamma(b - t).$$

Both functions solve $u'' = 0$. The function ϕ satisfies the left boundary condition, and ψ satisfies the right boundary condition. Their Wronskian is

$$\begin{aligned} W(t) &= \phi(t)\psi'(t) - \phi'(t)\psi(t) = -\gamma(\beta + \alpha(t - a)) \\ &\quad - \alpha(\delta + \gamma(b - t)) = -(\alpha\gamma L + \beta\gamma + \alpha\delta) = -D. \end{aligned}$$

Theorem 1 (Green's function). *Under assumptions (A1)–(A3), the Green's function for (3) is*

$$G(t, s) = \frac{1}{D} \begin{cases} \phi(t)\psi(s), & a \leq t \leq s \leq b, \\ \phi(s)\psi(t), & a \leq s \leq t \leq b. \end{cases} \quad (4)$$

Equivalently,

$$G(t, s) = \frac{\phi(\min\{t, s\})\psi(\max\{t, s\})}{D}.$$

Moreover, G is continuous and symmetric on $[a, b]^2$, and $G(t, s) \geq 0$.

Proof. For fixed s , the function $G(\cdot, s)$ is linear on either side of $t = s$, hence $G_{tt} = 0$ for $t \neq s$. The two branches agree at $t = s$. Their derivative jump is

$$G_t(s^+, s) - G_t(s^-, s) = \frac{\phi(s)\psi'(s) - \phi'(s)\psi(s)}{D} = \frac{W(s)}{D} = -1.$$

Thus $-G_{tt}(t, s) = \delta_s(t)$ in the distributional sense. The endpoint conditions follow because the left branch is a multiple of $\phi(t)$ at $t = a$, and the right branch is a multiple of $\psi(t)$ at $t = b$. Since $D > 0$, the associated homogeneous problem has only the zero solution, so the Green's function is unique. Finally, $\phi, \psi \geq 0$ on $[a, b]$ by (A2), proving nonnegativity. Symmetry follows directly from the minimum-maximum formula. $\square$

Theorem 2 (Green's representation). *If $f \in C[a, b]$, the unique solution u of (3) is*

$$u(t) = \int_a^b G(t, s)f(s)ds. \quad (5)$$

Consequently, every solution of (1) satisfies

$$x(t) = \int_a^b G(t, s)q(s)x(s)ds. \quad (6)$$

Proof. Let $v(t) = \int_a^b G(t, s)f(s) ds$. From (4),

$$v(t) = \frac{1}{D} \left[\psi(t) \int_a^t \phi(s)f(s)ds + \phi(t) \int_t^b \psi(s)f(s)ds \right].$$

Differentiation, using cancellation of the diagonal terms in the first derivative and the derivative jump in the second, gives $-v'' = f$. The endpoint conditions follow from those of ϕ and ψ . Uniqueness follows from $D > 0$. Taking $f = qx$ yields (6). $\square$

4 Lyapunov-type inequalities

Let

$$M_G := \max_{(t,s) \in [a,b]^2} G(t, s).$$

Continuity ensures that this maximum exists. Under (A2)–(A3), $M_G > 0$.

Theorem 3 (Green's-function Lyapunov inequality). *If (1) has a nontrivial solution, then*

$$\int_a^b |q(s)|ds \geq \frac{1}{M_G}. \quad (7)$$

A more informative necessary condition is

$$1 \leq \max_{t \in [a,b]} \int_a^b G(t, s)|q(s)|ds. \quad (8)$$

Proof. From (6) and $G \geq 0$,

$$\begin{aligned} |x(t)| &\leq \int_a^b G(t, s)|q(s)||x(s)|ds \\ &\leq \|x\|_\infty \int_a^b G(t, s)|q(s)|ds. \end{aligned}$$

Since x is nontrivial, $\|x\|_\infty > 0$. Choose $t_0 \in [a, b]$ with $|x(t_0)| = \|x\|_\infty$. Dividing the preceding inequality at $t = t_0$ by $\|x\|_\infty$ gives

$$1 \leq \int_a^b G(t_0, s)|q(s)|ds,$$

which proves (8). Since $G(t_0, s) \leq M_G$, we obtain

$$1 \leq M_G \int_a^b |q(s)|ds,$$

and hence (7). $\square$

Corollary 1 (Explicit Robin constant). *Under (A1)–(A3),*

$$M_G = \frac{1}{D} \max_{a \leq t \leq b} [\beta + \alpha(t - a)] [\delta + \gamma(b - t)]. \quad (9)$$

Therefore, every nontrivial solution satisfies

$$\int_a^b |q(s)|ds \geq \frac{D}{\max_{a \leq t \leq b} [\beta + \alpha(t - a)] [\delta + \gamma(b - t)]}. \quad (10)$$

Proof. Put $m = \min\{t, s\}$ and $r = \max\{t, s\}$. Since ϕ is nondecreasing and ψ is nonincreasing, $\phi(m)\psi(r) \leq \phi(r)\psi(r)$. Thus the maximum of G is attained on the diagonal, which gives (9). Substitution in (7) proves (10). $\square$

For an explicitly computable form, put $y = t - a$, so $0 \leq y \leq L$, and

$$C = \alpha\delta + \alpha\gamma L - \beta\gamma.$$

Then

$$DM_G = \max_{0 \leq y \leq L} \{\beta(\delta + \gamma L) + Cy - \alpha\gamma y^2\}.$$

If $\alpha\gamma > 0$, the maximizing point is

$$y_* = \min \left\{ L, \max \left\{ 0, \frac{C}{2\alpha\gamma} \right\} \right\}.$$

If $\alpha\gamma = 0$, the expression is linear or constant, so its maximum occurs at an endpoint unless it is constant.

Theorem 4 (Weighted eigenvalue estimate). *Let $w \in C[a, b]$ be nonnegative and not identically zero. If*

$$x''(t) + \lambda w(t)x(t) = 0$$

with the same Robin boundary conditions has a nontrivial solution, then

$$|\lambda| \geq \frac{1}{M_G \int_a^b w(s) ds}. \quad (11)$$

Proof. Apply Theorem 3 with $q(t) = \lambda w(t)$. Since $w \geq 0$,

$$\int_a^b |q(s)| ds = |\lambda| \int_a^b w(s) ds.$$

The stated estimate follows. $\square$

Theorem 5 (L^p -Lyapunov inequality). *Let $1 < p < \infty$ and $p' = p/(p - 1)$. If (1) has a nontrivial solution, then*

$$\|q\|_{L^p(a,b)} \geq \left[\max_{t \in [a,b]} \left(\int_a^b G(t, s)^{p'} ds \right)^{1/p'} \right]^{-1}. \quad (12)$$

Proof. Choose t_0 with $|x(t_0)| = \|x\|_\infty$. As in the proof of Theorem 3, using Hölder's inequality,

$$1 \leq \int_a^b G(t_0, s) |q(s)| ds \leq \left(\int_a^b G(t_0, s)^{p'} ds \right)^{1/p'} \|q\|_{L^p(a,b)}.$$

Bounding the first factor by its maximum over $t \in [a, b]$ proves (12). $\square$

5 Special cases

5.1 Dirichlet boundary conditions

Taking $\alpha = \gamma = 1$ and $\beta = \delta = 0$ gives $x(a) = x(b) = 0$, $D = L$, and

$$G(t, s) = \begin{cases} \frac{(t-a)(b-s)}{L}, & t \leq s, \\ \frac{(s-a)(b-t)}{L}, & s \leq t. \end{cases}$$

The diagonal maximum is $M_G = L/4$. Therefore Theorem 3 gives

$$\int_a^b |q(t)| dt \geq \frac{4}{L}.$$

The classical Dirichlet Lyapunov inequality is usually stated with strict inequality under its standard hypotheses; the norm argument above alone yields the non-strict version.

5.2 Dirichlet–Neumann conditions

For $x(a) = 0$ and $x'(b) = 0$, choose $\alpha = \delta = 1$ and $\beta = \gamma = 0$. Then $D = 1$ and $G(t, s) = \min\{t, s\} - a$, so $M_G = L$. Thus

$$\int_a^b |q(t)| dt \geq \frac{1}{L}.$$

5.3 Neumann–Dirichlet conditions

For $x'(a) = 0$ and $x(b) = 0$, choose $\beta = \gamma = 1$ and $\alpha = \delta = 0$. Then $G(t, s) = b - \max\{t, s\}$, $M_G = L$, and

$$\int_a^b |q(t)| dt \geq \frac{1}{L}.$$

Remark 1. For pure Neumann conditions, the auxiliary operator $-u''$ is resonant because constants belong to its kernel. In that case $D = 0$, so the Green's function above does not apply without a compatibility condition or a modified inverse.

6 Worked example

Consider

$$x''(t) + q(t)x(t) = 0, \quad 0 < t < 2, \quad x(0) = 0, \quad x'(2) = 0.$$

The Green's function for the auxiliary problem is $G(t, s) = \min\{t, s\}$, and $M_G = 2$. Hence any nontrivial solution must satisfy

$$\int_0^2 |q(t)| dt \geq \frac{1}{2}.$$

If $q(t) = \lambda$ is constant, this implies $|\lambda| \geq 1/4$.

For positive $\lambda = k^2$, the equation has solutions $x(t) = A \sin(kt)$. The boundary condition $x'(2) = 0$ requires $\cos(2k) = 0$, so

$$k = \frac{(2n+1)\pi}{4}, \quad \lambda_n = \frac{(2n+1)^2\pi^2}{16}, \quad n = 0, 1, 2, \dots$$

Thus the first positive eigenvalue is $\pi^2/16$, which is consistent with, but stronger than, the necessary bound $1/4$.

7 Discussion and research directions

The results above establish a reusable baseline for a broader study. Several directions require further investigation:

1. **Sharp constants.** Determine whether the Green's-function constant can be improved under symmetry, monotonicity, or sign assumptions on q , and establish whether equality is possible.
2. **General Robin parameters.** When some boundary coefficients are negative, positivity of G may fail. One can instead seek estimates based on $|G|$, while proving invertibility separately.
3. **Variable coefficients.** Study $-(p(t)x'(t))' = \lambda w(t)x(t)$ with $p(t) > 0$ and $w(t) \geq 0$.
4. **Nonlinear problems.** For $x'' + f(t, x) = 0$, additional growth or sign assumptions are needed; the linear integral representation cannot be used unchanged.
5. **Comparison with the literature.** Any proposed refinement should be compared with the existing literature, including the Robin inequalities and recent generalized-operator papers cited here.

8 Conclusion

We constructed the Green's function for a second-order equation with separated Robin boundary conditions under explicit nonresonance and positivity assumptions. The integral representation

yields an unweighted Lyapunov-type inequality, a weighted eigenvalue estimate, and an L^p -version. Dirichlet and mixed conditions are recovered as special cases, while pure Neumann conditions require separate treatment because of resonance.

The main necessary condition is

$$\int_a^b |q(t)|dt \geq \left(\max_{t,s \in [a,b]} G(t, s) \right)^{-1}.$$

This provides a rigorous starting point for research into sharper constants, optimality, and extensions to more general operators.

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