

The Norm of Elementary Operators: Numerical Range Bounds and the Hilbert Space Conjecture

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Abstract

This article surveys norm estimates for elementary operators, with particular emphasis on the Jordan elementary operator $U_{a,b}(x) = axb + bxa$ on $\mathcal{B}(\mathcal{H})$. We discuss the established inequality $\|U_{a,b}\| \geq \|a\| \|b\|$, which resolves Mathieu's conjecture, and examine the roles of finite-dimensional reduction, numerical-range geometry and completely bounded norm estimates in its proof. We distinguish numerical-range lower bounds from Haagerup-tensor upper bounds, and we separate the ordinary from the completely bounded norm, noting that the identification of the completely bounded norm with the Haagerup norm on prime C^* -algebras does not identify it with the ordinary norm. Explicit matrix examples, verified in full, demonstrate the sharpness of the universal lower constant and illustrate cases of strict inequality. We also discuss extensions to prime C^* -algebras and Hilbert modules, and describe the general variational norm formula of Timoney in terms of matrix numerical ranges and the tracial geometric mean.

Keywords: Elementary Operators; Numerical Range; Numerical Radius; Mathieu's Conjecture; Haagerup Tensor Norm; Completely Bounded Maps; Jordan Elementary Operator

1 Introduction and scope

Let $\mathcal{H}$ be a complex Hilbert space and $\mathcal{B}(\mathcal{H})$ the algebra of bounded linear operators on $\mathcal{H}$. An *elementary operator* is a map $\phi: \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ of the form

$$\phi(x) = \sum_{i=1}^n a_i x b_i, \quad a_i, b_i \in \mathcal{B}(\mathcal{H})$$

These operators arise naturally in the study of derivations, automorphisms and quantum channels, and they form the simplest non-trivial class of completely bounded maps on operator algebras. The norm problem asks for the determination of the norm of ϕ in terms of the coefficients.

Upper bounds are immediate from the triangle inequality and the Haagerup tensor norm,

$$\|\phi\| \leq \left\| \sum_i a_i \otimes b_i \right\|_h$$

where the subscript h denotes the Haagerup norm on the algebraic tensor product. Lower bounds, however, have constituted the central difficulty. The obstruction is structural: on commutative algebras, elementary operators can vanish despite non-zero coefficients, so no universal lower bound in terms of individual coefficient norms can exist. The problem becomes tractable precisely when the algebra is sufficiently non-commutative.

The focal point of this survey is the symmetrised two-sided multiplication

$$U_{a,b}(x) = axb + bxa$$

Mathieu [1, 2] conjectured in 1989 that

$$\|U_{a,b}\| \geq \|a\|\|b\|$$

for all a, b in $\mathcal{B}(\mathcal{H})$, and established the inequality with the constant $2/3$ in place of 1 [2]. This conjecture—the Hilbert space conjecture for elementary operators—remained open for over a decade. Partial progress included the self-adjoint case, settled by Stachó and Zalar [4]; the constant $2(\sqrt{2} - 1) \approx 0.8284$ in standard operator algebras [3, 8]; and the constant 1 under an additional hypothesis on the pencil $a + zb$ [5]. The conjecture was finally resolved in 2003–2004 by Timoney [9] and independently by Blanco, Boumazgour and Ransford [11].

This article is a survey and exposition of established results; an earlier short survey of the norm problem is that of Timoney [6]. We do not claim a new proof of Mathieu’s conjecture. Rather, we present the established theorem with precise attribution, explain the role of numerical range methods, provide fully verified examples, and delineate further directions. The central theorem is Theorem 6 of Timoney [9], which also extends the inequality to prime C^* -algebras, and it was obtained independently by Blanco, Boumazgour and Ransford [11].

The paper first establishes notation and preliminary results and then traces the development of the lower constants with verified attribution. It next presents the numerical range framework, states the main theorem with a faithful outline of the published proof, and analyses upper bounds via the Haagerup tensor norm. Extensions and sharpness are then discussed with explicit examples, further directions and questions are set out, and the paper concludes.

2 Preliminaries

Throughout, $\mathcal{H}$ denotes a complex Hilbert space, $\mathcal{B}(\mathcal{H})$ the algebra of bounded linear operators and $\mathcal{K}(\mathcal{H})$ the ideal of compact operators. For T in $\mathcal{B}(\mathcal{H})$, $\|T\|$ is the operator norm; when T is Hilbert–Schmidt, $\|T\|_2$ denotes the Hilbert–Schmidt norm.

2.1 Inner product and rank-one conventions

We take the inner product linear in its first variable. For ξ, η in $\mathcal{H}$, the rank-one operator $\xi \otimes \eta$ is defined by

$$(\xi \otimes \eta)z = \langle z, \eta \rangle \xi$$

and consequently

$$a(\xi \otimes \eta)b = (a\xi) \otimes (b^*\eta)$$

2.2 Numerical range

The numerical range of T in $\mathcal{B}(\mathcal{H})$ is

$$W(T) = \{\langle Tx, x \rangle : x \in \mathcal{H}, \|x\| = 1\}$$

and the numerical radius is $w(T) = \sup\{|\lambda| : \lambda \in W(T)\}$. The classical inequalities

$$w(T) \leq \|T\| \leq 2w(T)$$

hold in general, with equality $\|T\| = w(T)$ for normal T .

2.3 Operator and completely bounded norms

For a map $\phi: \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$, set

$$\|\phi\| = \sup_{\|x\| \leq 1} \|\phi(x)\|, \quad \|\phi\|_{cb} = \sup_{m \geq 1} \|\phi^{(m)}\|$$

where $\phi^{(m)}$ acts entrywise on $m \times m$ matrices over $\mathcal{B}(\mathcal{H})$. Always $\|\phi\| \leq \|\phi\|_{cb}$.

2.4 The Haagerup tensor norm

For $u = \sum_i a_i \otimes b_i$ in the algebraic tensor product $\mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{H})$, the Haagerup norm is

$$\|u\|_h = \inf \left\{ \left\| \sum_i u_i u_i^* \right\|^{1/2} \left\| \sum_i v_i^* v_i \right\|^{1/2} : u = \sum_i u_i \otimes v_i \right\}$$

The infimum runs over all representations of u ; consequently the value computed from a single displayed representation is only an upper estimate for $\|u\|_h$, and a matching lower bound must be established separately.

2.5 Elementary operators on norm ideals

For a norm ideal $J \subseteq \mathcal{B}(\mathcal{H})$ and tuples $A = (A_1, \dots, A_n)$, $B = (B_1, \dots, B_n)$, the elementary operator $R_{A,B}: J \rightarrow \mathcal{B}(\mathcal{H})$ is defined by

$$R_{A,B}(X) = \sum_{i=1}^n A_i X B_i$$

2.6 Numerical ranges for elementary operators

Following Seddik [7], for a Banach space E the spatial numerical range of T in $\mathcal{B}(E)$ is built from norming pairs,

$$V(T) = \{f(Tx) : f \in E^*, x \in E, \|f\| = \|x\| = f(x) = 1\}$$

while the algebraic numerical range is defined through normalised states,

$$W_{\text{alg}}(T) = \{F(T) : F \in \mathcal{B}(E)^*, \|F\| = F(I_E) = 1\}$$

The norming conditions are essential. Seddik's numerical-range paper [7] defines the algebraic numerical range through states, not through arbitrary four-vector coefficient expressions. For the symmetrised two-sided multiplication the state description yields

$$\begin{aligned} W_{\text{alg}}(U_{a,b}) \\ = \{ \langle a\xi, \xi \rangle \langle b\eta, \eta \rangle + \langle b\xi, \xi \rangle \langle a\eta, \eta \rangle : \|\xi\| = \|\eta\| = 1 \} \end{aligned}$$

3 Historical development of lower bounds

3.1 Chronology of the lower constants

The constants c in $\|U_{a,b}\| \geq c \|a\| \|b\|$ were obtained in the following order.

- **Mathieu (1989, 1990)**. The problem was posed and the inequality established with $c = 2/3$ [1, 2].

- **Stachó and Zalar (1996)**. The constant $c = 2(\sqrt{2} - 1) \approx 0.8284$ was obtained for standard operator algebras by Jordan triple system techniques [3]; the same constant was later reached by other routes, including Magajna and Turnšek (2003) [8].
- **Stachó and Zalar (1998)**. The conjecture was proved in full, with $c = 1$, for self-adjoint coefficients [4].
- **Barraa and Boumazgour (2001)**. The constant $c = 1$ was obtained under the additional hypothesis $\|a + zb\| \geq \|a\|$ for all z in $\mathbb{C}$ [5].
- **Timoney (2003) and Blanco–Boumazgour–Ransford (2004)**. The constant $c = 1$ was obtained in general, independently [9, 11].

Remark 1. Two cautions about this chronology are worth recording, since both occur in the secondary literature. First, the $2/3$ bound is Mathieu’s own and not a later improvement upon work of Stachó and Zalar; those authors strengthened the constant rather than initiating the study, and their 1996 paper precedes rather than follows the general Banach-space estimates often cited after it. Second, the contribution of Barraa and Boumazgour is the conditional constant 1 under a hypothesis on the pencil $a + zb$, not the constant $2/3$.

Remark 2 (Triangle-inequality arguments give no positive constant). A natural first attempt writes $U_{a,b}(x)$ as a sum of two terms P and Q and applies

$$\|P + Q\| \geq \left| \|P\| - \|Q\| \right|$$

With $\|P\|, \|Q\| \leq 1$ this yields no positive uniform lower bound, since $P = -Q$ is compatible with the hypotheses. For the same reason, the function

$$f(t) = \sqrt{1 - t^2} + t\sqrt{2 - t^2} - t^2$$

which sometimes appears in discussions of the constant $2(\sqrt{2} - 1)$, satisfies $f(0) = 1$, so $2(\sqrt{2} - 1)$ is not its maximum. Note also that the constant is $2(\sqrt{2} - 1) \approx 0.8284$ and not $2\sqrt{2} - 1 \approx 1.8284$; a constant exceeding 1 would contradict the sharpness examples given below. The published derivations of the $2/3$ and $2(\sqrt{2} - 1)$ constants proceed quite differently, through Jordan triple identities and numerical-range geometry rather than through a norm comparison of the two summands; see [2] and [3].

3.2 The self-adjoint case

Stachó and Zalar [4] confirmed Mathieu’s conjecture for self-adjoint coefficients.

Theorem 1 (Stachó–Zalar [4]). *Let a, b in $\mathcal{B}(\mathcal{H})$ be self-adjoint. Then $\|U_{a,b}\| \geq \|a\| \|b\|$.*

The self-adjoint hypothesis places the numerical ranges $W(a)$ and $W(b)$ on the real line and makes the spectral theorem available,

$$a = \int \lambda dE_a(\lambda), \quad b = \int \mu dE_b(\mu)$$

for real spectral measures E_a, E_b .

Remark 3. Two points require care in the spectral argument. First, for self-adjoint a the spectral value of modulus $\|a\|$ need not be $+\|a\|$; the operator $a = -I$ is a counterexample. The correct statement is that there exists λ in the spectrum of a with $|\lambda| = \|a\|$ together with unit vectors ξ for which $\|a\xi - \lambda\xi\| \rightarrow 0$; one should not assert $\langle a\xi, \xi \rangle \approx \|a\|$. Second, self-adjointness makes the diagonal values

$\langle a\xi, \xi \rangle$ real, but it does not make the off-diagonal matrix coefficients $\langle a\xi, \eta \rangle$ real. It is precisely the reality of the relevant coefficients that blocks the phase cancellation obstructing the general case, so the self-adjoint result locates the difficulty of the full conjecture in the non-normality of the coefficients.

3.3 The role of numerical range methods

Seddik's 2002 paper [7] provides a systematic framework connecting numerical ranges and norms of elementary operators; it is distinct from his 2004 paper [12]. Seddik proves an inclusion of the form

$$\text{co} \left\{ \sum_{i=1}^n \langle A_i \xi, \xi \rangle \langle B_i \eta, \eta \rangle : \|\xi\| = \|\eta\| = 1 \right\} \subseteq W_{\text{alg}}(R_{A,B})$$

and consequently

$$\|R_{A,B}\| \geq \sup_{\|\xi\|=\|\eta\|=1} \left| \sum_{i=1}^n \langle A_i \xi, \xi \rangle \langle B_i \eta, \eta \rangle \right|$$

Here the direction of the argument is sound: a concrete coefficient-generated set lies inside an algebraic numerical range whose radius is bounded by the operator norm. Seddik also obtains equality in the corresponding numerical-range inclusion for generalised derivations.

Remark 4 (Direction of the inclusion argument). The inclusion

$$V(R_{A,B}) \subseteq W_{\text{alg}}(R_{A,B})$$

gives only

$$\sup_{\lambda \in V(R_{A,B})} |\lambda| \leq \sup_{\lambda \in W_{\text{alg}}(R_{A,B})} |\lambda|$$

Knowing that a norm dominates the radius of a smaller set does not show that it dominates the radius of a larger one. The valid inference is

$$\|R_{A,B}\| \geq \sup \{ |\lambda| : \lambda \in V(R_{A,B}) \}$$

so any lower bound on the numerical radius of $V(R_{A,B})$ is automatically a lower bound on $\|R_{A,B}\|$. In Seddik's inclusion the coefficient-generated set is the smaller one, which is why that argument runs in the required direction.

The three-step structure—define the two numerical ranges, prove the inclusion, invoke norm domination—applies to derivations, generalised derivations, two-sided multiplications and their symmetrisations. For $U_{a,b}$ the algebraic numerical range is the set displayed in the preliminaries, and the remaining task is to show that its radius is at least $\|a\| \|b\|$. That estimate alone does not establish Mathieu's conjecture; a further argument is required, and it is supplied by the proofs of the main theorem discussed below.

4 The numerical range framework

4.1 Rank-one test operators

For $X = \xi \otimes \eta$ with $\|\xi\| = \|\eta\| = 1$,

$$U_{a,b}(\xi \otimes \eta) = (a\xi) \otimes (b^* \eta) + (b\xi) \otimes (a^* \eta)$$

The Hilbert–Schmidt norm of X is 1, and $\|U_{a,b}\| \geq \|U_{a,b}(X)\|$. Taking the Hilbert–Schmidt inner product with a second rank-one operator gives the matrix-coefficient identity

$$\begin{aligned} \langle U_{a,b}(\xi \otimes \eta), \eta' \otimes \xi' \rangle &= \langle a\xi, \eta' \rangle \langle b\xi', \eta \rangle \\ &+ \langle b\xi, \eta' \rangle \langle a\xi', \eta \rangle \end{aligned}$$

The norms of the two summands are

$$\|(a\xi) \otimes (b^* \eta)\| = \|a\xi\| \|b^* \eta\|$$

and *not* $\|a\xi\| \|b\eta\|$; the latter substitution is false in general. For unit vectors the identity above yields

$$|\langle a\xi, \eta' \rangle \langle b\xi', \eta \rangle + \langle b\xi, \eta' \rangle \langle a\xi', \eta \rangle| \leq \|U_{a,b}\|$$

directly, so the four-vector expression has a legitimate role as a rank-one matrix-coefficient test, independent of any numerical-range inclusion. What such a test does not by itself deliver is the product lower bound.

Remark 5 (Approximate norm attainment does not imply alignment). Choosing ξ and η with $\|a\xi\| \approx \|a\|$ and $\|b\eta\| \approx \|b\|$ does not permit one also to arrange

$$\eta \approx \frac{a\xi}{\|a\xi\|}, \quad \xi \approx \frac{b\eta}{\|b\eta\|}$$

These are additional compatibility requirements, and an argument based on rank-one tests must establish them rather than assume them. For the same reason a phase adjustment cannot be treated as automatically making the two summands reinforce one another: they share the vectors ξ, η, ξ', η' , so their phases are not independently adjustable. The proofs of the main theorem outlined below avoid the difficulty altogether by reducing to a finite-dimensional problem.

4.2 The Hilbert–Schmidt numerical radius

The Hilbert–Schmidt class $C_2(\mathcal{H})$ is itself a Hilbert space, and for T in $C_2(\mathcal{H})$ one has

$$w(T) \leq \|T\| \leq \|T\|_2$$

Remark 6 (Two different norms). The quantity $\|U_{a,b}\|$ is defined by

$$\|U_{a,b}\| = \sup_{\|X\| \leq 1} \|U_{a,b}(X)\|$$

using the operator norm on both sides, and its definition gives no estimate for a Hilbert–Schmidt output norm. In particular $\|U_{a,b}\| < 1$ does not imply $\|U_{a,b}(T)\|_2 < 1$ for $\|T\|_2 = 1$. The immediately available inference is

$$\|U_{a,b}(T)\| \leq \|U_{a,b}\| \|T\| \leq \|U_{a,b}\| \|T\|_2$$

If the Hilbert–Schmidt restriction is to be used it should carry its own notation,

$$\|U_{a,b}\|_{2 \rightarrow 2} = \sup_{\|T\|_2 \leq 1} \|U_{a,b}(T)\|_2$$

for which the directly available estimate is

$$\|U_{a,b}(T)\|_2 \leq 2\|a\|\|b\|\|T\|_2$$

4.3 Correct use of the inclusion

The inclusion of Remark 4 permits the inference $\|R_{A,B}\| \geq \sup\{|\lambda| : \lambda \in V(R_{A,B})\}$, and nothing stronger. If one shows that the algebraic numerical range contains elements of modulus arbitrarily close to a target value c , a lower bound $\|R_{A,B}\| \geq c$ follows only once one proves separately that such elements lie in $V(R_{A,B})$ or in its closure. Restricting attention to the values actually realised inside V states what must be proved; it does not prove it.

5 The main theorem

5.1 Statement

Theorem 2 (Timoney [9, Theorem 6]; Blanco–Boumazgour–Ransford [11]). *Let $\mathcal{H}$ be a complex Hilbert space and a, b in $\mathcal{B}(\mathcal{H})$. Then $\|U_{a,b}\| \geq \|a\| \|b\|$.*

This confirms Mathieu's conjecture [1, 2]. Timoney's Theorem 6 also gives the inequality when A is a prime C^* -algebra and a, b lie in its multiplier algebra; his proof identifies $\mathcal{H} = \mathbb{C}^2$ as the essential case. The article of Blanco, Boumazgour and Ransford establishes the Hilbert-space inequality independently.

5.2 Outline of Timoney's proof

We state the theorem as an established result and give an explicitly labelled outline of the published argument rather than a reconstruction of it. Timoney's proof proceeds in five stages.

1. **Reduction to a two-dimensional problem.** The general case is reduced to 2×2 matrices.
2. **A Haagerup-tensor decomposition and cb-norm estimates.** The completely bounded norm is related to the Haagerup tensor norm of the coefficient tensor.
3. **A norm and cb-norm formula for certain symmetric 2×2 matrices.** Explicit formulas are established for both norms of these matrices.
4. **Unitary transformations and symmetrisation.** Unitary conjugation is used to normalise the structure.
5. **Convex geometry of a joint numerical range.** The final step uses geometric properties of the joint numerical range, including an ellipse–hyperbola argument.

Hilbert–Schmidt norms occur in the finite-dimensional estimates of stage 3, but the argument is not a Hilbert–Schmidt numerical-radius contradiction; by Remark 6 no such contradiction is available from the definition of $\|U_{a,b}\|$.

5.3 The completely bounded norm

For the related symmetric operator

$$V_{a,b}(x) = a^*xb + b^*xa$$

Magajna [10] computed the norm and showed that it coincides with the completely bounded norm.

Theorem 3 (Magajna [10]). *For all a, b in $\mathcal{B}(\mathcal{H})$, $\|V_{a,b}\| = \|V_{a,b}\|_{cb}$. Moreover $\|V_{a,b}\|$ equals the norm of the restriction of $V_{a,b}$ to the self-adjoint part of $\mathcal{B}(\mathcal{H})$, and*

$$\begin{aligned} \|V_{a,b}\| &= \inf \left\{ \left\| pa^*a + rb^*b - qi(a^*b - b^*a) \right\| : p, q, r \in \mathbb{R}, p, r > 0, pr - q^2 = 1 \right\} \end{aligned}$$

The result extends to any prime C^* -algebra. Magajna also notes explicitly that the corresponding statement for the Jordan elementary operator $U_{a,b}(x) = axb + bxa$ is not available: there the norm and the completely bounded norm can differ.

Remark 7 (Positivity). The identity underlying the symmetric case is

$$V_{a,b}(x) = \frac{1}{2} [(a+b)^*x(a+b) - (a-b)^*x(a-b)]$$

This exhibits $V_{a,b}$ as a difference of completely positive maps, not a sum, and the map need not be positive: $a = I, b = -I$ gives $V_{a,b}(x) = -2x$. Magajna's equality is a genuine theorem, but it is not a consequence of positivity.

Remark 8 (Primeness does not identify the two norms). Two statements should be kept apart. The inequality

$$\|\phi\| \leq \|\phi\|_{cb}$$

holds always, while in the prime C^* -algebra setting Mathieu's theorem gives

$$\|\phi\|_{cb} = \left\| \sum_{i=1}^n a_i \otimes b_i \right\|_h$$

The second does not imply $\|\phi\| = \|\phi\|_{cb}$; it identifies the completely bounded norm with the Haagerup norm and says nothing about the ordinary norm. Timoney [9] discusses elementary operators on $\mathcal{B}(\mathcal{H})$ for which the two norms differ, while Magajna [10] proves equality for the particular form $V_{a,b}$.

Case 1: prime C^* -algebras. Let A be a prime C^* -algebra and ϕ an elementary operator on A . Mathieu's theorem states that $\|\phi\|_{cb}$ equals the Haagerup norm of the coefficient tensor. The proof shows that the canonical map from the Haagerup tensor product $A \otimes_h A$ into the space of completely bounded maps on A is isometric when A is prime; primeness excludes the non-trivial two-sided ideals that would allow the completely bounded norm to fall below the Haagerup norm. For the ordinary norm one retains only $\|\phi\| \leq \|\phi\|_{cb}$.

Case 2: the Calkin algebra. Let $C(\mathcal{H}) = \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ be the Calkin algebra and let ϕ be the elementary operator on $C(\mathcal{H})$ induced by coefficients in $\mathcal{B}(\mathcal{H})$. When discussing the induced operator one should work with the quotient coefficients

$$\dot{a}_i = a_i + \mathcal{K}(\mathcal{H}), \quad \dot{b}_i = b_i + \mathcal{K}(\mathcal{H})$$

Equality of the ordinary and completely bounded norms for elementary operators on the Calkin algebra is supported in the literature, but it does not follow merely from primeness together with a rich supply of projections, and the Haagerup norm is not computed through the Fredholm index. The relevant broader setting is that of antiliminal C^* -algebras, which Timoney [9] discusses.

6 Upper bounds and the Haagerup tensor norm

6.1 The Haagerup norm

For $\phi(x) = \sum_i a_i x b_i$ the Haagerup tensor norm of the coefficient tensor $u = \sum_i a_i \otimes b_i$ is the quantity displayed in the preliminaries, and the fundamental chain of inequalities is

$$\|\phi\| \leq \|\phi\|_{cb} \leq \|u\|_h$$

For prime C^* -algebras Mathieu's theorem gives $\|\phi\|_{cb} = \|u\|_h$. For $\mathcal{B}(\mathcal{H})$ the operator and completely bounded norms of an elementary operator can differ, but the Haagerup norm remains an upper bound for both.

Example 1 (An exact case). Let $\mathcal{H}$ be a separable infinite-dimensional Hilbert space and let $\phi(x) = axb$ with a, b non-zero. The coefficient tensor is $a \otimes b$, with

$$\|a \otimes b\|_h = \|a\| \|b\|$$

and the two-sided multiplication $L_a R_b$ has norm $\|a\| \|b\|$. Hence $\|\phi\| = \|\phi\|_{cb} = \|u\|_h = \|a\| \|b\|$ and the Haagerup bound is exact.

Remark 9. Claims that the Haagerup norm takes a given value “under suitable orthogonality conditions on the coefficients” require care in two respects. First, the conditions must specify which of the ranges of a_i, a_i^*, b_i and b_i^* are involved; range orthogonality of the first family alone is not enough. Second, as noted in the preliminaries, the Haagerup norm is an infimum over representations, so the value read off a single displayed representation is in general only an upper estimate. Example 1 and Example 2 are therefore stated with explicit coefficients and with separate proofs of both norms.

6.2 The gap between the operator and completely bounded norms

For elementary operators on $\mathcal{B}(\mathcal{H})$ the two norms need not agree. Magajna [10] records that the norm and completely bounded norm of $U_{a,b}$ may differ, while they do coincide in several identifiable situations: when a and b are commuting normal operators; for the symmetric 2×2 case treated in [9, Proposition 4]; and for the symmetric operator $V_{a,b}$ of Theorem 3. Combining Theorem 2 with the chain above gives

$$\|a\| \|b\| \leq \|U_{a,b}\| \leq \|U_{a,b}\|_{cb} \leq \|a \otimes b + b \otimes a\|_h$$

so for the symmetrised two-sided multiplication all three quantities are trapped between $\|a\| \|b\|$ and the Haagerup norm of the coefficient tensor. What is not available is a quantitative comparison of $\|U_{a,b}\|_{cb}$ with $\|U_{a,b}\|$ in terms of the coefficients, nor one in terms of the number of summands of a general elementary operator; see Problem 3 below.

6.3 The Haagerup norm for the symmetrised operator

For $U_{a,b}(x) = axb + bxa$ the coefficient tensor is $a \otimes b + b \otimes a$, and

$$\|a \otimes b + b \otimes a\|_h \geq \|a\| \|b\|$$

with equality in favourable cases. Theorem 2 shows that the operator norm satisfies the same lower bound, so the Haagerup norm and the operator norm agree in their lower estimates for symmetrised two-sided multiplications.

Example 2 (A symmetrised operator on $\mathbb{C}^2$). Let $\mathcal{H} = \mathbb{C}^2$ with standard basis e_1, e_2 , and let

$$a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

so that $\|a\| = \|b\| = 1$. Writing e_{ij} for the matrix units, $ab = e_{11}$ and $ba = e_{22}$; in particular $ab \neq ba$ and $ab + ba = I$. For $x = (x_{ij})$,

$$U_{a,b}(x) = axb + bxa = \begin{pmatrix} x_{22} & 0 \\ 0 & x_{11} \end{pmatrix} = \text{diag}(x_{22}, x_{11})$$

whence

$$\|U_{a,b}(x)\| = \max(|x_{11}|, |x_{22}|) \leq \|x\|$$

Taking $x = I$ gives equality, so

$$\|U_{a,b}\| = 1 = \|a\|\|b\|$$

The Haagerup norm is also 1. The representation $u_1 = a, v_1 = b, u_2 = b, v_2 = a$ gives

$$aa^* + bb^* = I, \quad b^*b + a^*a = I$$

whence

$$\|a \otimes b + b \otimes a\|_h \leq 1$$

and the reverse inequality follows from $\|U_{a,b}\| = 1$ together with the fundamental chain of inequalities for the Haagerup norm given above.

7 Extensions and sharpness

7.1 Standard operator algebras

In a standard operator algebra on a Banach space the optimal constant is not known to be 1. The bound

$$\|U_{a,b}\| \geq 2(\sqrt{2} - 1)\|a\|\|b\|$$

holds in that generality [3, 8], while the stronger bound $\|U_{a,b}\| \geq \|a\|\|b\|$ is available under an additional hypothesis on the pencil $a + zb$, namely

$$\|a + zb\| \geq \|a\| \text{ for all } z \in \mathbb{C}$$

equivalently $\inf_{\lambda} \|a + \lambda b\| = \|a\|$ [5]. The loss of the optimal constant in the Banach space setting reflects the failure of the parallelogram law and the absence of an inner product structure.

7.2 Prime C*-algebras and Hilbert modules

Arambašić and Gogić [14] extend Mathieu's theorem to Hilbert modules. For a right Hilbert A -module X with its canonical operator space structure, an elementary operator on X is a map $\phi(x) = \sum_i u_i x v_i$ with the u_i in $\mathcal{B}(X)$ and the v_i in the multiplier algebra $M(A)$; when $X = A$ this agrees with the usual notion. Their theorem is that for a non-zero Hilbert A -module X the completely bounded norm of every elementary operator on X agrees with the Haagerup norm of the corresponding tensor in $\mathcal{B}(X) \otimes M(A)$ if and only if A is prime. The equivalence, rather than a one-directional implication, is the content of the result, and the hypothesis that X be non-zero should be retained.

7.3 Sharpness of the constant

The constant 1 in Theorem 2 is optimal as soon as $\dim \mathcal{H} \geq 2$, and the cleanest witnesses are orthogonal projections.

Example 3 (Equality). Let p and q be non-zero projections with $pq = qp = 0$. Writing $r = p + q$ and $s = p - q$, both contractions, one has

$$U_{p,q}(x) = \frac{1}{2} [(p+q)x(p+q) - (p-q)x(p-q)]$$

and therefore $\|U_{p,q}\| \leq 1$. Choosing unit vectors ξ in the range of p and η in the range of q and taking $x = \xi \otimes \eta$ gives $U_{p,q}(x) = x$, so

$$\|U_{p,q}\| = 1 = \|p\|\|q\|$$

Magajna [10] likewise identifies non-zero orthogonal projections as an equality example.

Strict inequality also occurs, so it is Example 3 and not the following examples that establishes optimality of the constant.

Example 4 (Strict inequality). (i) For $a = b = I$,

$$U_{I,I}(x) = 2x, \|U_{I,I}\| = 2 = 2\|I\|\|I\|$$

(ii) For $a = b$ a positive contraction with $\|a\| = 1$,

$$U_{a,a}(x) = 2axa, \|U_{a,a}\| = 2\|a\|^2 = 2$$

In both cases the inequality of Theorem 2 is strict.

Remark 10 (Dimension one). On a one-dimensional complex Hilbert space every operator is a scalar and

$$\|U_{a,b}\| = 2\|a\|\|b\|$$

so the constant 1 is not attained there; the sharpness statement requires $\dim \mathcal{H} \geq 2$.

We do not assert that equality is the exception rather than the rule. Without a precise measure-theoretic or topological reading, and a result supporting it, such a statement carries no content. What the examples establish is that equality and strict inequality both occur, and that the constant 1 cannot be improved.

7.4 The tracial geometric mean approach

Timoney [13] gives a general variational formula for the norm of an elementary operator $T(x) = \sum_j a_j x b_j$ with ℓ summands on $\mathcal{B}(\mathcal{H})$, in which a is treated as a row and b as a column of operators. Writing

$$Q(b, \eta) = (\langle b_i^* b_j \eta, \eta \rangle)_{i,j=1}^\ell$$

for a column tuple b and a unit vector η , and defining the tracial geometric mean of positive semidefinite $\ell \times \ell$ matrices by

$$\text{tgm}(X, Y) = \text{Tr} \sqrt{\sqrt{X} Y \sqrt{X}}$$

the formula reads

$$\|T\| = \sup \{ \text{tgm}(Q(a^*, \xi), Q(b, \eta)) : \xi, \eta \in \mathcal{H}, \|\xi\| = \|\eta\| = 1 \}$$

An equivalent form in terms of the trace-class norm is

$$\|T\| = \sup \left\{ \left\| \sqrt{Q(b, \eta)^t a^*} \right\|_{S_1} : \eta \in \mathcal{H}, \|\eta\| = 1 \right\}$$

The same paper extends the treatment to general C^* -algebras and characterises compactness of elementary operators.

Three matters should be distinguished here: an exact variational formula, which exists; a simple, readily evaluated coefficient formula, which may remain difficult; and efficient computation together with structural classification, which are separate problems again. General variational formulas are available through matrix numerical ranges and the tracial geometric mean, but their effective evaluation and simplification for important coefficient classes remain substantial problems.

8 Further directions and questions

The questions below are offered as directions rather than as a catalogue of open problems; each should be checked against the current literature before being treated as open.

Problem 1. Evaluate

$$\left\| \sum_{i=1}^n a_i x b_i \right\|$$

for $n \geq 3$ when the coefficients are not symmetrised. Exact variational formulas are available through matrix numerical ranges and the tracial geometric mean (see the tracial geometric mean approach above); what is missing is their effective evaluation and simplification for important coefficient classes. The Haagerup norm provides an upper bound and rank-one test operators provide lower bounds.

Problem 2. Characterise the elementary operators for which

$$\|\phi\| = \|\phi\|_{cb}$$

Mathieu's theorem identifies the completely bounded norm with the Haagerup norm on prime C^* -algebras, and Magajna's result establishes equality of the two norms for

$$x \mapsto a^* x b + b^* x a$$

The general characterisation is not known. As Remark 8 notes, the prime-algebra theorem does not by itself settle any instance of this question.

Problem 3. Is there a quantitative comparison of the two norms for an elementary operator of length ℓ —a bound

$$\|\phi\|_{cb} \leq C_\ell \|\phi\|$$

with C_ℓ depending only on ℓ —and if so, what is the optimal constant? For $\ell = 2$ the symmetrised case is well understood (Theorem 2 and Theorem 3), but we are not aware of examples that would settle optimality, and we do not assert that none exists.

Problem 4. Characterise equality in Theorem 2. Orthogonal projections give equality (Example 3) and $a = b = I$ gives strict inequality (Example 4); a complete characterisation of the equality case is not known.

Problem 5. Extend the numerical range approach to elementary operators on non-prime C^* -algebras and on Banach spaces lacking a Hilbert space structure. The constant $2(\sqrt{2} - 1)$ for standard operator algebras on Banach spaces is not known to be optimal.

Problem 6. Develop explicit finite-dimensional examples and numerical methods for computing norms of elementary operators, in particular an effective evaluation of the variational formula of Section VII-D.

9 Conclusion

The resolution of Mathieu's conjecture for the symmetrised two-sided multiplication on $\mathcal{B}(\mathcal{H})$ is a landmark in the theory of elementary operators. The proof—by finite-dimensional reduction, Haagerup-tensor decomposition, completely bounded norm estimates and the convex geometry of joint numerical ranges—shows the force of combining operator-algebraic and geometric techniques. The constant 1 in $\|U_{a,b}\| \geq \|a\| \|b\|$ is optimal for $\dim \mathcal{H} \geq 2$, as orthogonal projections show.

The numerical-range framework systematised by Seddik [7, 12] provides a conceptual setting that transcends the particular problem of $U_{a,b}$. The inclusion of the coefficient-generated set in the algebraic numerical range of an elementary operator relates its action on a norm ideal to the individual coefficients, and the norm of the operator dominates the numerical radius of that action. Used in the direction set out in Remark 4, the framework is likely to remain useful in attacking the questions that remain.

The contrast between the complete solution for symmetrised two-sided multiplications and the fragmentary state of knowledge for general elementary operators is striking. Exact variational formulas are available through matrix numerical ranges and the tracial geometric mean, but their effective evaluation and simplification for important coefficient classes remain substantial problems. A complete solution to the norm problem on $\mathcal{B}(\mathcal{H})$ would be a major advance in operator theory, with implications for quantum information theory—finite-dimensional quantum channels admit elementary-operator representations of Kraus form, although a Kraus representation carries a complete positivity constraint that a general elementary operator does not—and for the classification of derivations on C^* -algebras.

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