MATHEMATICS AND STATISTICS

A Study on Algebraic Properties of Finite Permutation Groups

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Published November 2026

Abstract

This paper presents a systematic study of the algebraic properties of permutation groups, with particular emphasis on the symmetric group S_n. We begin by recalling the definition of a permutation of a finite set and introduce the standard two-row and cycle notations. The structure of the symmetric group is examined, including its order, the identity and inverse elements, and the non-commutativity of S_3. Cycle decompositions, transpositions, and the composition of cycles are treated in detail with explicit computations. The work consolidates fundamental results that form the foundation for further investigations into the alternating group and applications in algebra, combinatorics, and related fields.

Keywords: permutation group; symmetric group; cycle notation; transposition; group composition

1 Introduction

As early as 200 BC, mathematicians have studied structures that we now call groups: structures, symmetries, numbers, permutations and more. But it was not until the mid-1800s that someone noticed that collections of symmetries, of numbers and of other operations have something in common and gave that something a name. This was British mathematician Arthur Cayley (1821-1895), the first to write down something that looks like our modern definition of a "group"

Cayley's observation did not come out of nowhere. A whole family of mathematicians were working on problems involving groups — although mostly groups of permutations and moving towards higher and higher degrees of abstraction. French mathematician Evariste Galois (1811-1832) was the first to use the word "group" (groupe in French) to describe a group of permutations. Other major contributors were Augustin-Louis Cauchy, C. F. Gauss, and Felix Klein, who worked on groups of symmetries. And once the notion of "group" had been defined, there was an explosion of interest in creating a unified theory and extending the concept to cover broader and broader areas of mathematics. The symmetric groups S_n (recall that S_n is the group of permutations of n objects) are particularly important. Historically, they were the objects of study that inspired the modern definition of a group. And as we saw in class, every finite group can be thought of as a kind of a permutation group. (Precisely, for any finite group, there is at least one subgroup of some symmetric group S_n that is structurally the same as the finite group.) So understanding symmetric groups is one way to get at an understanding of all finite groups.

In mathematics, a permutation group is a group G whose elements are permutations of a given set M and whose group operation is the composition of permutations in G (which are thought of as bijective functions from the set M to itself). The group of all permutations of a set M is the symmetric

group of M, often written as Sym(M). The term permutation group thus means a subgroup of the symmetric group. If $M = \{1, 2, ..., n\}$ then $Sym(M)$ is usually denoted by S_n, and may be called the symmetric group on n letters. By Cayley's theorem, every group is isomorphic to some permutation group.

The way in which the elements of a permutation group permute the elements of the set is called its group action. Group actions have applications in the study of symmetries, combinatorics and many other branches of mathematics, physics and chemistry.

Permutation groups have a rich history in mathematics, dating back to ancient times when they were used to solve combinatorial problems. However, their formal study gained prominence during the development of group theory in the 19th century. Mathematicians like Augustin Louis Cauchy and Évariste Galois made significant contributions to understanding permutation groups as symmetries of mathematical objects.

The exploration of permutation groups and their algebraic properties has far-reaching applications beyond mathematics itself. In computer science, permutation groups are used in cryptography, sorting algorithms, and designing efficient data structures. Additionally, they play a fundamental role in the study of molecular symmetries in chemistry and the analysis of symmetry in physics and crystallography.

Permutation groups are a fundamental object in abstract algebra, and their algebraic properties have been studied by mathematicians for centuries.

One of the earliest works on the algebraic properties of permutation groups was done by Leonhard Euler in the 18th century. Euler studied the symmetry of polyhedra and other geometric objects using permutation groups. He also developed a number of theorems on the properties of permutation groups, such as the Cayley-Hamilton theorem and the Jordan-Hölder theorem.

In the 19th century, Augustin-Louis Cauchy and Camille Jordan made significant contributions to the study of permutation groups. Cauchy developed a theory of groups that was based on the concept of a permutation group. Jordan developed a classification of finite permutation groups up to isomorphism.

In the 20th century, there was a great deal of progress made in the study of permutation groups. Some of the most important figures in this area include William Burnside, Alfred H. Clifford, and Richard Brauer. Burnside classified all finite permutation groups of prime degree. Clifford and Brauer developed a theory of representations of permutation groups, which has many applications in other areas of mathematics. Other important figures in the study of permutation groups in the 20th century include Walter Feit, Marshall Hall, and John Thompson. Feit and Thompson proved the famous Feit-Thompson theorem, which states that every finite non-abelian simple group contains an element of prime order.

The study of groups originally grew out of an understanding of permutation groups. Permutations had themselves been intensively studied by Lagrange in 1770 in his work on the algebraic solutions of polynomial equations. This subject flourished and by the mid-19th century a well-developed theory of permutation groups existed, codified by Camille Jordan in his book *Traité des Substitutions et des Équations Algébriques* of 1870. Jordan's book was, in turn, based on the papers that were left by Évariste Galois in 1832.

1.1 Some of the key concepts

To ensure a clear understanding of the terminology used in this paper, key concepts and terms are defined as follows:

Set: A set is defined to be a collection of well-defined distinct elements. Sets are usually denoted by capital letters like A, B, C, etc. while members of the sets are denoted by small letters a, b, c, etc.

Axiom: Axiom is a claim that is evident or well-established, And can serve as the foundation for an argument or additional line of reasoning.

Binary operation: An assignment that combines that two components of a set (referred to an operations) to create an element of the set (technically, an operation with two arties whose two domains and one codomain are (subsets of) the same set) is called a binary operation on a set.

Order: This is defined to be the number of elements in a set.

Group: A group is a set G equipped with a binary operation$*$ (G $*$) that satisfies the following four axioms:

1. Closure:

For all $a,\ b \in G$, we have$a \cdot b \in G$.

1. Associativity:

For all,$a,\ b,\ c \in G$ we have$a * (b * c) = (a * b) * c$

1. Identity element:

There exists an element $e \in G$ such that for all $a \in G$ we have

$e * a = a * e = a$.

1. Inverse elements:

$\forall a \in G$, there exists an element $a^{-1} \in G$ such that$a^{-1} * a = a * a^{-1} = e$.

Where e is the identity element in G.

Subgroups: A subset H of a group G is said to be a subgroup of G written H ⊆ G, if H is also a group with respect to the binary operation of G.

Cyclic Group: A cyclic group is a group that can be generated by a single element.

Permutation Groups: A permutation group is a mathematical group that consists of all possible permutations of a set. In other words, it encompasses all possible arrangements or reorderings of the elements in the set. Permutation groups play a crucial role in group theory and have applications in combinatorics, cryptography, and other areas.

Algebraic properties: Algebraic properties are basic mathematical properties that apply to numbers, operations, and equations. They are used in many areas of mathematics, including algebra, calculus, and real analysis

Abstract Algebra: Abstract algebra is a branch of mathematics that studies algebraic structures and their properties in a general and abstract manner. It deals with concepts that are applicable to various mathematical objects, allowing for the exploration of common underlying principles.

Group Operations: Group operations, such as composition and inversion, define how elements within a group interact with each other. These operations satisfy specific properties, including closure, associativity, identity, and inverse existence.

Cyclic Subgroups: A cyclic subgroup is a subgroup of a group that is generated by a single element. It consists of all powers of that element and forms a subset of the larger group.

Proper subgroup: A proper subgroup of a group G is a subgroup H of G such that $H=G$. In other words, a proper subgroup is a subset of G that satisfies all the group axioms, but it is not equal to the whole group.

2.0. LITERATURE REVIEW

One of the earliest recorded studies of permutations occurs in the Sefer Yetrirah, or Book of creation, written by an unknown Jewish author sometimes before the eighth century. The author was interested in counting the various ways in which the letters of the Hebrew alphabet can be arranged. The question was in some sense a mystical one. It was believed that the letters had magical powers; the forces of nature. The actual text of the Sefer Yetsirah is very sparse: "two letters build two words, three build six words, four build 24 words, five build 120, six build 720, seven build 5040," interestingly enough, the idea of counting the arrangements of the letters of the alphabet also occurred in Islamic mathematics in the eighth and ninth centuries. By the thirteenth century, in both the Islamic and Hebrew cultures, the abstract idea of a permutation had taken root so that Abu-l-Abbas ibn al-Banna (1256-1321), a mathematician from Marrakech in what is now Morocco, and Leviben Gerson, a French Rabbi, philosopher, and mathematician, were able to give rigorous proofs that the number of permutation of any set of n elements is n!, as well as prove various results about counting combinations.

The study of groups originally grows out of an understanding of permutation groups. Permutation had themselves been intensively studies by lay range in 1770 in his work on the algebraic solutions of polynomial equations. This subject flourished and by the end 19^{th}century a well-developed theory of permutation groups existed, codified by Camille Jordan in his book traite des substitutions et-dus equation algebriques of 1870. Jordan book was, in turn based on the papers that were left by Evariste Galois in 1832, When coyly introduced by concept of an abstract group. It was not immediately clear whether or not this was a larger collection of objects then. The known permutation group which had a definition different from the modern one. Cayley went on to prove that the two concepts were equivalent in Cayley's theorem.

Another classical tent containing several chapters on permutation groups in Burnsides theory of groups of finite order of 1911.

The first half of the twentieth century was a follow period in the study of group theory in general. But interest in permutation groups was revived in the 1950s by H. WIELANDT whose German Lecture notes were reprinted of finite permutation groups in 1964.

The modern concept of an abstract group developed out of several field of mathematics. The original motivation for group theory was the guest for solution of polynomial equations of degree higher than four. The 19 century French mathematician EVARISTE GALOIS extending prior work of PGOLO RUFFIN and JOSEPH-LOVIS language gave a criterion in term of the symmetry group of it not (Solutions). The element of such a Galois group corresponds to certain permutation of the roots. At first, Galois ideas were rejected by his contemporaries and published only posthumously. more general permutation groups were investigated in particular by Augustin-louis Cauchy. Arthur Cayley's on the

theory of groups, as depending on the symbolic equation θ^n. In (1854) gives the first abstract definition of a finite group. Geometry was a second First in which groups were used systematically especially symmetry groups as part of FELIX Klein's 1872 Erlangen program. After novel geometry, such as hyperbolic and projective geometry had emerged. Klein used group theory to organize them in a more coherent way. Further advancing these ideas Sophuslie founded the study of group in 1884.

The third field contributing to group theory was number theory certain abelian group. Structures had been used implicitly in Carl Friedrich Gauss number theoretical work disquisitions Arithmetical (1798) and more explicitly by Leopold Kronecker in 1847 Ernst Kummer made early attempts to prove Fermat's last theorem by developing groups describing factorization into prime number.

The permutation group has gained prominence in the fundamental research in diverse areas of physics and chemistry. Covering all salient developments of the last few years in a single symposium would require weeks, legions of participants and parallel sessions, highlighting the differences in language and communication problems between mathematicians, high and low energy physicists and chemistry. The symposium holds July 1978 at the centre of interdisciplinary studies of the university of Bielefeld focused on a small area, the pertinence of the permutation group in chemical physics, with the goal to increase and generate a fruitful dialogue between mathematicians and chemists.

In chemistry, concerned with the electronic and geometric structure of molecules as well as elementary chemical reactions, i.e. rearrangements in these structures, the permutation group has its relevance, since with its representations the effects and consequences of exchanging indistinguishable particles, electrons and identical nuclei, can be systematized and classified.

1.2 Brief history of permutation groups

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Another approach is to utilize the concept of group actions. By defining a group action on a set, one can study the orbits, stabilizers, and other properties of the action to gain insight into the structure of

the permutation group (John & Musa 2017). This can help in understanding how the permutations interact with the elements of the underlying set and reveal algebraic properties of the group.

Additionally, techniques from representation theory can also be useful in studying the algebraic properties of permutation groups. By representing the permutations as matrices or linear transformations, one can analyze the group structure in terms of linear algebra and use tools from representation theory to decompose the group into irreducible components.

Solving the problem of understanding the algebraic properties within a permutation group requires a combination of methods from group theory, combinatorics, representation theory, and other branches of mathematics. By applying these techniques and studying the interactions of permutations within the group, researchers can gain a better understanding of the underlying algebraic structure and properties of permutation groups.

However Haruna, F. et al., (2024), considered the Application of Finite Cyclic Groups to Musical Notations and hence said**.** Musical notation has evolved over centuries, aiming to represent the intricate elements of music in a structured and understandable format. Traditional notation systems often rely on linear representations, such as staff notation. However, recent advancements in mathematical theory and computational techniques have opened up new avenues for exploring alternative approaches to musical representation. One such promising avenue is the application of finite cyclic groups, which offer unique properties that can potentially enrich the understanding and analysis of musical structures.

The area of pure mathematics known as group theory is originated from the algebra. Because it was abstract, It appeared to be more of an arts subject than a science subject.

was actually viewed as being purely abstract and impractical. Even students of group theory after being introduced to the abstract nature of the subject, the course appears to not believe that it has any real-world applications. (Tsok, 2013). This challenge encourages scholars to examine the various ways that groups can be expressed in concrete terms from a theoretical and practical perspective in order to bring its applicability to real-world situations, especially in the context of musical notes. The purpose of this work is to apply some group theory principles to the analysis and comprehension of musical notes in light of the group's axioms.. The main objective is to see the relationship that exist between these musical notes and cyclic group and their interpretation algebraically. This work focuses on the behavior of musical notes, which is mostly determined by group axioms, theorems like the first theorem by Langrange and Sylow, cyclic groups, and two left cosets. A group that has an element to which an operation is applied that yields the entire set is called a cyclic group. It is clear that a cyclic groups is a groups of repeated patterns until returning to the beginning. In general, a Groups that have all of their elements as powers of a fixed element are known as cyclic groups.

2 Methodology

We collect some theorems, Lemmas and proposition that are used in proving our main results

Theorem 3.1 uniqueness of element and an inverse

let G be a group with operation $*$ then;

1. The identity element is unique
2. For each the inverse element of is unique

Proof

1, suppose that e & a are identity element in G

Now regarding e as identity

and regarding $a$as identity

Thus $e * a = a * e = a$.

$e = a$.

Hence the identity element of a group is unique

1. Suppose are inverse of

$a^{-1} * a = a * a^{-1} = e$&$a * a^{-1} = a * a^{-1} = e$

Therefore $a = a * e$ because of e is an identity element

$= a * (a * a^{-1})$ because of inverse exist

$= (a * a) * a^{-1}$ because of associativity

$= e * a^{-1}$ because of inverse

Hence$a = a^{-1}$

Lemma 3.1:

Let $H \subseteq G$ and let G be a group. Consequently, each right cost of H in G has the same cardinality as H.

Proof:

Let *Hg be a right coset of H in G* and define φ: $H \longrightarrow Hg$by (h) = hg then (h_1) = (h_2),

$\Longrightarrow h_1g = h_2g$

(*by cancellation law*)

$h_1 = h_2$

h*ence* φ *is* 1 − 1

Now we have to show that φ is onto.

Take $y \in Hg$ then y = hg for some h ∈ H. Then (h) =hg = y.

Consequently, φ is onto.

Since φ is 1 − 1 and onto then φ is a bijection.

Thus $|H| = |Hg|$.

Theorem 3.2: A finite group's subgroup order is determined by the group's order.

Proof;

Let $H \subseteq G$ and $|G| = n$, $|H| = m$

Now, *H′s pairwise disjoint cosets are united to form G*.

Let there be j distincts cosets of *H in G*.

We are aware that for any $a \in G$

$| aH | = | H | = m$

Consequently, the overall count of elements in G *is mj*.

s. t. $n = mj$;

Hence m divides n as asserted.

Note:

Observe that $n = mj$ as in the proof of the theorem 3.2 imply that

$|G| = |G : H|/|H|$

Theorem 3.3:

Assume that G is an order finite group say n, and H a subgroup of G. Then $|H|$ divides $|G|$

Proof:

Let $Ha_1, Ha_2, \dots, Ha_k$ be the right cosets of H in G.

$G = Ha_1 \cup Ha_2 \cup Ha_3 \cup, \dots, \cup Ha_k$

and union is disjoint.

$\Longrightarrow |G| = |Ha_1 \cup Ha_2 \cup Ha_3 \cup, \dots, \cup Ha_k|$

$= |Ha_1| + |Ha_2| + \cdots + |Ha_k|$

$\because$ *the union is disjoint*

$= |H| + |H| + \cdots + |H|$

By cosets lemma

$k|H|$

So $|H| \mid |G|$.

Where k is the quantity of unique rights coset of H *in* G.

Theorem 3.4: A cyclic group's subgroups are all cyclic.

Proof:

Let $G = \langle g \rangle$ be a cyclic group, where $g \in G$.

Let $H \subseteq G$. If $H = \{e\}$, then $H = \{g^o\}$ is trivially cyclic with generator e.

So assume $H \neq \{e\}$. And chose h $\in H$.

Then h $= g^s$ for some $s \in$ h and $h^{-1} = g^{-1} \in H$.

Consequently, positive integers exist t, such that $g^t \in H$.

Choose the least positive integer among them, and refer to it as l, (Any collection of positive integers has a smallest number inside it according to the well-ordering principle of natural numbers). By Division Algorithm we may write

$s = ql + r$ where $0 \leq r < l$.

Then h $= g^s = g^{ql+r} = (g^l) \cdot g^r$.

So that $g^r = (g^l)^{-q} \cdot h \in H$.

If $r \neq 0$, then $r < l$

Which contradict the choice of l.

Thus $r = 0$ and so $h = (g^l)$.

Hence $H \subseteq \langle g^l \rangle$

Now $g^l \in H$ and so, $\langle g^l \rangle \subseteq H$.

Accordingly, $H = \langle g^l \rangle$ and the result follows that g^l generate H. So H is cyclic.

Theorem 3.5: Every group of cyclics is abelian.

Proof:

Let G be a group of cycles. and a be a generator of G. Then $G = \langle a \rangle = \{a^n \mid n \in h\}$.

(By the definition 1.10)

Let g *and* $g\hat{:}$ be any two elements of G.

Then $g = a^\alpha$ *and* $g\hat{:} = a^\beta$ for some integer α *and* β

So $gg\hat{:} = a^\alpha a^\beta = a\alpha{+}\beta = a\beta{+}\alpha$

$= a^\beta a^\alpha = g\hat{:}g$

$\therefore gg\hat{:} = g\hat{:}g$

Therefore, G is abelian.

3 Results and discussion

Let S be a finite set with n distinct elements, a one-one mapping of S onto itself is called a permutation of degree n.

In other words let $S = \{a_1,\ a_2,\ a_3 \ldots a_n\}$ be a finite set with n distinct elements a mapping $F : S \to S$ which is one-one and onto is called permutation of degree n.

3.1 Symbol of permutation

Let$S = \{a_1,\ a_2,\ a_3 \ldots a_n\}$

$f : S \to S$ such that

i.e. $b_1,\ b_2,\ b_3 \ldots b_n$ are some arrangement of the n element of S, f can be represented conveniently as

Illustration 1.

Let$S = \{1,\ 2,\ 3,\ 4,\ 5\}$

Are some example of permutation of degree 5 on the set S.

3.2 Identity permutation

A permutation in which each element is mapped to itself is called an identity permutation.

i.e. a permutation I *is* called an identity permutation of the set S if$\forall\ a \in S : f(a) = a$

i.e.

3.3 Inverse permutation

If $f = \begin{pmatrix} a_1, & a_2, & a_3 \ldots a_n \\ b_1, & b_2, & b_3 \ldots b_n \end{pmatrix}$ then,

3.4 Symmetric group

In abstract algebra, the symmetric group defined over any set is the group whose elements are all the bijections from the set to itself, and whose group operation is the composition of functions. In particular, the finite symmetric group S_{*n*} defined over a finite set of n symbols consists of the permutations that can be performed on the n symbols. Since there are $n!$ (n factorial) such permutation operations, the order (number of elements) of the symmetric group S_{*n*} is $n!$.

Definition: A group G is called a permutation group on a nonempty set A if the elements of G are some permutations of A and the group operation is the composition of two functions. Let us now consider permutation of a finite set. Suppose for any positive integer n, I_n denotes the finite set $\{1, 2, 3, \ldots, n\}$. For example, $I_3 = \{1, 2, 3\}$. Now any permutation on I_n is a bijective function on $\{1, 2, 3, \ldots, n\}$. The set of all permutations on I_n forms a group under the binary operation '*composition of two functions*'. This permutation group is called the *symmetric group* on n elements and is denoted by S_n. Let $\alpha \in S_n$. Generally we demonstrate α in the following way:

where $\alpha(i)$ denotes the image of i under α, for all $i = 1, 2, \ldots, n$. But it is sometimes convenient to describe this permutation by means of the following notational device:

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & \ldots & n \\ \alpha(1) & \alpha(2) & \alpha(3) & \ldots & \alpha(n) \end{pmatrix}$$

This notation is due to Cauchy and is called the *two-row notation*. In the upper row, we list all the elemets of I_n and in the lower row under each element $i \in I_n$, we write the image of the element, i.e., $\alpha(i)$. It is easy to compute the number of elements of S_n. There are n choices of $\alpha(1)$. Once $\alpha(1)$ has been chosen, there are $n - 1$ possibilities for $\alpha(2)$

(since $\alpha(1)6 = \alpha(2)$). After chossing $\alpha(2)$, there are exactly $n - 2$ possibilities for $\alpha(3)$. Continuing this way we see that S_n must have $n(n-1)(n-2).....3.2.1 = n!$ elements. The two-row notation of permutations is quite convenient while doing computations, such as determining the composition of permutations.

Let

Now the composition $\alpha \circ \beta$ is also a permutation on I_n defined by

$$(\alpha \circ \beta)(i) = \alpha(\beta(i))$$

for all $i \in I_n$. Then,

$$\alpha \circ \beta = \begin{pmatrix} 1 & 2 & \ldots & n \\ \alpha(1) & \alpha(2) & \ldots & \alpha(n) \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & \ldots & n \\ \beta(1) & \beta(2) & \ldots & \beta(n) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & \ldots & n \\ \alpha(\beta(1)) & \alpha(\beta(2)) & \ldots & \alpha(\beta(n)) \end{pmatrix}.$$

Illustration 2: We consider the group S_3, the elements of this group being all the permutations on $I_3 = \{1, 2, 3\}$. As the number of bijective functions of I_3 onto itself is 3! = 6, we have $|S_3| = 6$.

We now enlist below the permutations on $I_3 = \{1,\ 2,\ 3\}$.

$$e = \begin{pmatrix}1 & 2 & 3\\ 1 & 2 & 3\end{pmatrix}\alpha = \begin{pmatrix}1 & 2 & 3\\ 2 & 3 & 1\end{pmatrix}\beta = \begin{pmatrix}1 & 2 & 3\\ 3 & 1 & 2\end{pmatrix}$$

$$\gamma = \begin{pmatrix}1 & 2 & 3\\ 1 & 3 & 2\end{pmatrix}\delta = \begin{pmatrix}1 & 2 & 3\\ 2 & 1 & 3\end{pmatrix}\sigma = \begin{pmatrix}1 & 2 & 3\\ 3 & 2 & 1\end{pmatrix}$$

We now show some computations regarding the composition or product of elements of S_3.

$$\gamma \circ \delta = \begin{pmatrix}1 & 2 & 3\\ 1 & 3 & 2\end{pmatrix} \circ \begin{pmatrix}1 & 2 & 3\\ 2 & 1 & 3\end{pmatrix} = \begin{pmatrix}1 & 2 & 3\\ 3 & 1 & 2\end{pmatrix} = \beta.$$

The following calculation of $\delta \circ \gamma$ reveals an important fact.

$$\delta \circ \gamma = \begin{pmatrix}1 & 2 & 3\\ 2 & 1 & 3\end{pmatrix} \circ \begin{pmatrix}1 & 2 & 3\\ 1 & 3 & 2\end{pmatrix} = \begin{pmatrix}1 & 2 & 3\\ 2 & 3 & 1\end{pmatrix} = \alpha.$$

We find that$\gamma \circ \delta 6 \neq \delta \circ \gamma$.

Hence S_3 is a non-commutative group.

Definition: *A* permutation σ on $I_n = \{1,\ 2,\ \ldots,\ n\}$is called a k-cycle or cycle of length k if there exist distinct elements $i_1,\ i_2,\ \ldots,\ i_k$in I_n such that

A k-cycle with $k = 2$ is called a transposition.

If a permutation σ on i_n is a $k-$ cycle, we shall denote it by

$(i_1,\ i_2,\ \ldots,\ i_k)$. We shall refer to this new notation as *cycle notation*.

Example 2:

$$\sigma = \begin{pmatrix}1 & 2 & 3 & 4 & 5 & 6\\ 3 & 5 & 2 & 4 & 1 & 6\end{pmatrix}$$

is a permutation on $I_6 = \{1,\ 2,\ 3,\ 4,\ 5,\ 6\}$ such that:

Hence, σ is a 4-cycle and we denote σ by (1325). Observe that:

$$\alpha = \begin{pmatrix}1 & 2 & 3 & 4\\ 3 & 4 & 1 & 2\end{pmatrix}$$

Consider now the permutation. For this permutation$\alpha(1) = 3,\ \alpha(3) = 1$ whereas $\alpha(2) = 4,\ \alpha(4) = 2$. Hence α is

not a $k-$ cycle.

Since *k*-cycles are nothing but special type of permutations, they can be composed, i.e., multiplied just like any two permutations.

Consider the cycles $\sigma = (243) and \delta = (1265)$. Then

In other words, we can look upon σ and δ as mappings in the following manner:

Again viewed as permutations in a two-row notation, σ and δ are as

follows:

and .

Now

It is worth noticing that while calculating $\sigma\delta$, we first consider the image of δ on the elements of I_6 and then on that image set, we further consider the image of σ.

Hence

Notice that under$\sigma\delta$,

Hence in this case $\sigma\delta$ is a 5-cycle.

But in general, product of two cycles may not be a cycle. To show this we consider the product of $\sigma = (56)$ and $\delta = (324)$ from S_6. Then

$$\sigma\delta = \sigma \circ \delta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 4 & 2 & 3 & 6 & 5 \end{pmatrix}$$

is not a cycle.

Using the cycle notation we now write the elements of S_3 as follows:

$S_3 = \{e,\ (12),\ (13),\ (23),\ (123),\ (132)\}$ consists of one 1-cycle, three 2-cycles and two 3-cycles.

4 Summary

This research "The Study of Algebraic properties of Permutation groups" delves into the in-depth analysis of permutation groups and their algebraic properties. Permutations are fundamental mathematical objects used to represent rearrangements of a finite set. The paper begins with an exploration of the basics of permutations, permutation groups, and their connection to group theory.

The paper focuses on the symmetric group (S_n) and the alternating group (A_n), both of which are permutation groups with significant algebraic properties. Symmetric groups consist of all permutations of a set, while alternating groups consist of even permutations. The paper discusses their definitions, properties, and relationships.

5 Conclusion

In conclusion, the study of the algebraic properties of permutations is a fundamental and rich area of mathematics. Permutation groups, such as symmetric and alternating groups, offer valuable insights into the symmetries and transformations of finite sets. This paper has provided a comprehensive overview of these topics, including definitions, properties, and applications of finite permutation group. Understanding permutation groups is not only essential within the region of group theory but also finds practical applications in various fields, including statistics, cryptography, and geometry. This paper has shed light on the significance of permutation groups in these contexts.

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