NUMBER THEORY

Arithmetic Partitions of Natural Numbers, Zeta Regularization, Prime Divisibility and Dirichlet L-Functions: A Mathematical Study of Finite and Infinite Partitions

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Abstract

The partition of the positive integers into arithmetic progressions provides a useful framework for studying divergent series through analytic continuation. This paper investigates arithmetic partitions of the natural numbers and their relationship with the Riemann zeta function, Hurwitz zeta function, Bernoulli polynomials, prime divisibility and Dirichlet L-functions. The ordinary series of positive integers diverges, but its zeta-regularized value is $\zeta(-1) = -1/12$. We derive formulas for regularized sums of residue classes modulo an integer, examine partitions determined by prime divisibility, and obtain weighted identities involving Dirichlet characters. We extend the method to higher powers and discuss an infinite partition based on powers of two. Particular attention is given to the distinction between finite additivity and countable additivity in regularized summation.

Keywords: Arithmetic partitions; Riemann zeta function; Hurwitz zeta function; zeta regularization; Bernoulli polynomials; prime numbers; Dirichlet characters; Dirichlet L-functions

1 Introduction

The positive integers $\mathbb{N} = \{1, 2, 3, \ldots\}$ can be divided into disjoint subsets according to congruence classes and divisibility conditions. For example, $\mathbb{N}$ is the disjoint union of the odd positive integers and the even positive integers. The ordinary series $1 + 2 + 3 + \cdots$ diverges because its partial sums tend to infinity. However, analytic continuation of the Riemann zeta function gives $\zeta(-1) = -1/12$ [1, 2]. This is a regularized value, not an ordinary sum.

This paper studies how zeta regularization behaves under finite arithmetic partitions. The Hurwitz zeta function and Bernoulli polynomials provide explicit formulas for residue classes. Prime divisibility and Dirichlet characters then give further arithmetic decompositions.

2 Preliminaries

Definition 2.1. (Riemann zeta function) For $\mathrm{Re}(s) > 1$, the Riemann zeta function is $\zeta(s) = \sum_{n\geq 1} n^{-s}$. It has a meromorphic continuation to the complex plane with a simple pole at $s = 1$.

Definition 2.2. (Hurwitz zeta function) For $a > 0$ and $\mathrm{Re}(s) > 1$, the Hurwitz zeta function is $\zeta(s, a) = \sum_{n\geq 0} (n + a)^{-s}$. It also admits analytic continuation in s.

Definition 2.3. (Bernoulli polynomials) The Bernoulli polynomials are defined by the generating function

$$\frac{te^{xt}}{e^t - 1} = \sum_{n\geq 0} B_n(x)\frac{t^n}{n!} \tag{1}$$

In particular, $B_2(x) = x^2 - x + 1/6$.

Theorem 2.1. (Hurwitz zeta values at nonpositive integers) For every integer $m \geq 0$,

$$\zeta(-m,\ a) = -\frac{B_{m+1}(a)}{m+1} \tag{2}$$

In particular, $\zeta(-1, a) = -B_2(a)/2$. This standard identity connects special values of the Hurwitz zeta function with Bernoulli polynomials [1, 3].

3 Finite partitions into residue classes

Fix a positive integer q. For $r = 1, \ldots, q$, define $A_r = \{qn + r : n = 0, 1, 2, \ldots\}$. These are the q residue classes represented by 1 through q.

Theorem 3.1. (Residue-class partition theorem) The sets $A_1, \ldots, A_q$ form a partition of the positive integers.

Proof. By the division algorithm, every positive integer n has a unique representation $n = qt + r$ with $t \geq 0$ and $1 \leq r \leq q$. Hence every n belongs to exactly one A_r. The sets are pairwise disjoint and their union is $\mathbb{N}$. □

Definition 3.1. (Zeta-regularized sum of a residue class) Define the regularized sum of A_r by analytic continuation as

$$S_r := q\zeta\left(-1,\ \frac{r}{q}\right) = -\frac{q}{2}B_2\left(\frac{r}{q}\right) \tag{3}$$

Expanding B_2 gives the useful formula

$$S_r = -\frac{r^2}{2q} + \frac{r}{2} - \frac{q}{12} \tag{4}$$

Theorem 3.2. (Finite residue-class identity) For every positive integer q,

$$\sum_{r=1}^{q} S_r = \zeta(-1) = -\frac{1}{12} \tag{5}$$

Proof. The Hurwitz zeta multiplication identity [3] states $\sum_{1\leq r\leq q} \zeta(s, r/q) = q^s\zeta(s)$. At $s = -1$, multiplication by q gives $\sum_r q\zeta(-1, r/q) = \zeta(-1)$. By the definition of S_r, the left side is $\sum_r S_r$. □

Remark. This is a finite identity between analytically continued values. It does not assert ordinary convergence of $1 + 2 + 3 + \cdots$.

4 Explicit examples

Example 4.1. (Modulus 2) For $q = 2$, $B_2(1/2) = -1/12$ and $B_2(1) = 1/6$. Therefore

$$S_1 = -B_2\left(\frac{1}{2}\right) = \frac{1}{12},\ S_2 = -B_2(1) = -\frac{1}{6} \tag{6}$$

Thus $S_1 + S_2 = 1/12 - 1/6 = -1/12$. The first class is the odd positive integers and the second is the even positive integers.

Example 4.2. (Modulus 3) For $q = 3$, the three residue-class sums are

$$S_1 = \frac{1}{12},\ S_2 = \frac{1}{12},\ S_3 = -\frac{1}{4} \tag{7}$$

Their sum is $1/12 + 1/12 - 1/4 = -1/12$.

Example 4.3. (Modulus 5) For $q = 5$, direct substitution into $S_r = -(5/2)B_2(r/5)$ gives

$$S_1 = -\frac{1}{60},\ S_2 = \frac{11}{60},\ S_3 = \frac{11}{60},\ S_4 = -\frac{1}{60},\ S_5 = -\frac{5}{12} \tag{8}$$

Adding these five values gives $-1/12$.

5 Partitions determined by prime divisibility

Let p be prime. Put $M_p = \{p, 2p, 3p, \ldots\}$ and $U_p = \{n \geq 1 : p \text{ does not divide } n\}$. These sets form a partition of the positive integers.

Theorem 5.1. (Regularized sum of multiples of p) The zeta-regularized sum of the positive multiples of p is

$$\operatorname{Reg}\sum_{n\geq 1} pn = p\zeta(-1) = -\frac{p}{12} \tag{9}$$

Proof. For $\operatorname{Re}(s) > 1$, $\sum_{n\geq 1}(pn)^{-s} = p^{-s}\zeta(s)$. Analytic continuation to $s = -1$ gives $p\zeta(-1) = -p/12$. □

Theorem 5.2. (Regularized sum of nonmultiples of p) The Dirichlet series over integers not divisible by p is $(1 - p^{-s})\zeta(s)$. At $s = -1$ its continued value is $(1 - p)\zeta(-1) = (p - 1)/12$. Thus the two classes have regularized values $-p/12$ and $(p - 1)/12$, whose sum is $-1/12$.

Example 5.1. (Some prime values) Table 1 lists the two regularized values for the first five primes.

Table 1. Regularized sums of the multiples and nonmultiples of a prime p.

Prime p	Multiples of p	Nonmultiples of p
2	−1/6	1/12
3	−1/4	1/6
5	−5/12	1/3
7	−7/12	1/2
11	−11/12	5/6

6 Simultaneous divisibility by two primes

Consider divisibility by 2 and 3. The positive integers split into four disjoint classes: divisible by neither; divisible by 2 but not 3; divisible by 3 but not 2; and divisible by both. These classes correspond modulo 6 to residues {1, 5}, {2, 4}, {3} and {6}, respectively.

Theorem 6.1. (Four-class partition identity) For modulus 6, the six individual residue-class values are

$$S_1 = -\frac{1}{12},\ S_2 = \frac{1}{6},\ S_3 = \frac{1}{4},\ S_4 = \frac{1}{6},\ S_5 = -\frac{1}{12},\ S_6 = -\frac{1}{2} \tag{10}$$

Consequently, the four divisibility classes have regularized sums −1/6, 1/3, 1/4 and −1/2. Their total is

$$-\frac{1}{6}+\frac{1}{3}+\frac{1}{4}-\frac{1}{2}=-\frac{1}{12} \tag{11}$$

Proof. Add the corresponding residue-class values: $S_1 + S_5 = -1/6$, $S_2 + S_4 = 1/3$, $S_3 = 1/4$ and $S_6 = -1/2$. Their sum is $-1/12$. □

7 Dirichlet characters and L-functions

Definition 7.1. (Dirichlet character) A Dirichlet character modulo q is a periodic, completely multiplicative arithmetic function χ, with $\chi(n + q) = \chi(n)$, and $\chi(n) = 0$ whenever $\gcd(n, q) > 1$.

Definition 7.2. (Dirichlet L-function) For $\mathrm{Re}(s) > 1$, $L(s, \chi) = \sum_{n\geq 1} \chi(n)n^{-s}$. Grouping the terms by residue classes gives [1]

$$L(s,\ \chi)=q^{-s}\sum_{r=1}^{q}\chi(r)\zeta\left(s,\ \frac{r}{q}\right) \tag{12}$$

At $s = -1$, this becomes

$$L(-1,\ \chi)=\sum_{r=1}^{q}\chi(r)S_r \tag{13}$$

Thus the special value $L(-1, \chi)$ is a character-weighted combination of the regularized residue-class sums.

Example 7.1. (Nonprincipal character modulo 3) Take $\chi(1) = 1, \chi(2) = -1$ and $\chi(3) = 0$. Since $S_1 = S_2 = 1/12$,

$$L(-1,\ \chi)=S_1-S_2=0 \tag{14}$$

This is consistent with the parity rule for Dirichlet L-functions [4]: the character modulo 3 is odd, giving a trivial zero at this negative integer.

Example 7.2. (Quadratic character modulo 5) The quadratic character modulo 5 has values $\chi(1) = 1$, $\chi(2) = -1, \chi(3) = -1, \chi(4) = 1$ and $\chi(5) = 0$. Hence

$$L(-1,\ \chi)=S_1-S_2-S_3+S_4=-\frac{1}{60}-\frac{11}{60}-\frac{11}{60}-\frac{1}{60}=-\frac{2}{5} \tag{15}$$

8 Higher powers

For a nonnegative integer m, the zeta-regularized power sum is defined by $\zeta(-m)$. For example, $\zeta(0) = -1/2$, $\zeta(-1) = -1/12$, $\zeta(-2) = 0$ and $\zeta(-3) = 1/120$ [2, 5]. These are regularized values, not ordinary convergent sums.

Definition 8.1. (Regularized power sum in a residue class) For $1 \leq r \leq q$, define

$$S_r^{(m)} := q^m\zeta\left(-m,\ \frac{r}{q}\right)=-\frac{q^m B_{m+1}\left(\frac{r}{q}\right)}{m+1} \tag{16}$$

Theorem 8.1. (Higher-power partition identity) For every integer $m \geq 0$,

$$\sum_{r=1}^{q} S_r^{(m)} = \zeta(-m) \tag{17}$$

Proof. Apply the Hurwitz multiplication identity $\sum_r \zeta(s, r/q) = q^s\zeta(s)$, set $s = -m$, and multiply by q^m. The result follows. □

Example 8.1. (Cubic powers) Since $\zeta(-3) = 1/120$, the regularized sums of the cubes in all q residue classes add to $1/120$. The individual terms are expressed through Bernoulli polynomials of degree four.

9 Infinite partitions and countable additivity

For $j = 0, 1, 2, \ldots$, define $C_j = \{2^j(2n + 1) : n = 0, 1, 2, \ldots\}$. Every positive integer has a unique representation $n = 2^j(2t + 1)$, where $2t + 1$ is odd. Thus the sets C_j form a countable partition of the positive integers.

Theorem 9.1. (Regularized sum of each dyadic class) The regularized sum of C_j is

$$S(C_j) = \frac{2^j}{12} \tag{18}$$

Proof. The regularized sum of the positive odd integers is $1/12$. Scaling every term by 2^j scales the regularized value by 2^j. Hence $S(C_j) = 2^j/12$. □

The formal sum over all j is $\sum_{j\geq 0} 2^j/12$, which diverges. This does not contradict the finite residue-class identity. Analytic continuation establishes the finite identities used above, but it does not automatically give countable additivity over arbitrary infinite partitions. Any infinite rearrangement therefore requires a separately specified and justified regularization procedure.

10 Main observations

1. Every positive integer belongs to exactly one residue class modulo q.
2. The regularized sum of a residue class is expressed through a Bernoulli polynomial.
3. The sum of the q residue-class values is $-1/12$ for the first-power case.
4. Prime divisibility gives regularized values $-p/12$ and $(p - 1)/12$ for multiples and nonmultiples.
5. Simultaneous divisibility conditions can be studied using residue classes modulo a product of distinct primes.
6. Dirichlet L-values at negative integers are weighted combinations of arithmetic-progression regularized sums.
7. Higher powers lead to identities involving higher Bernoulli polynomials.
8. Countable partitions require care because regularized summation is not generally countably additive.

11 Further research directions

The principal identities in this paper are established results from analytic number theory. The residue-class formula follows from the Hurwitz zeta function, while the prime-divisibility identities follow from the Euler factor decomposition of the zeta function. The following topics are possible directions for further original research, subject to a careful literature review.

11.1 Partitions associated with several primes

Let $P = \{p_1, \ldots, p_k\}$ be a finite set of distinct primes and $Q = p_1 p_2 \cdots p_k$. The Chinese Remainder Theorem identifies divisibility patterns with residue classes modulo Q. One research objective is to obtain compact formulas for the regularized sum of each divisibility pattern as a function of the selected primes.

11.2 Character-weighted partitions

The identity $L(-m, \chi) = \sum_r \chi(r) S_r^{(m)}$ can be studied for different m and characters. The parity of a character and the vanishing of special L-values provide a systematic avenue for investigation.

11.3 Higher Bernoulli polynomial identities

The values $-q^m B_{m+1}(r/q)/(m+1)$ can be examined for symmetry, sign patterns, divisibility properties and sums over selected residue classes.

11.4 Infinite partitions

A rigorous classification of infinite partitions for which a specified regularization is compatible with a chosen summation operation is an important problem. The regularization method and the conditions permitting rearrangement must be stated and proved.

11.5 Comparison of regularization methods

Zeta regularization may be compared with Abel summation, Cesàro summation where applicable, and suitable cutoff or finite-part methods. These methods should not be assumed equivalent without proof.

12 Conclusion

Arithmetic partitions connect elementary divisibility properties with analytic number theory. The Hurwitz zeta function expresses the regularized sum of each residue class in terms of Bernoulli polynomials, and the finite sum of these values reproduces the corresponding Riemann zeta value. Prime divisibility gives simple formulas, while simultaneous divisibility can be studied using the Chinese Remainder Theorem. Dirichlet characters connect weighted residue classes with Dirichlet L-functions, and the method extends to higher powers.

The infinite dyadic partition illustrates a limitation: finite additivity of these regularized identities does not imply countable additivity. This distinction is essential for a mathematically correct treatment of divergent series. The framework offers a unified setting for studying arithmetic progressions, prime-divisibility patterns, Bernoulli polynomials and special values of zeta and L-functions.

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