FUNCTIONAL ANALYSIS AND DIRICHLET SERIES

A Note on Topological Properties of a Space of a Class of Functions Represented by Dirichlet Series

Kirti Chauhan

Department of Mathematics, P.P.N. College, Kanpur, India

Published November 2026

Abstract

In this paper, we have defined a class of functions represented by generalized Dirichlet series whose coefficients satisfy certain given conditions. The region of convergence of the generalized Dirichlet series is obtained depending upon the fixed Dirichlet series having the same exponent sequence. This class is complete and becomes a Hilbert space. This class is topologically well-behaved with the convergence in the topology of space implying uniform convergence of the corresponding functions on every compact subset.

Keywords: Dirichlet series; abscissa of convergence; abscissa of absolute convergence; norm; parallelogram law

1 Introduction

The Dirichlet series has a long history that dates back to the nineteenth century. In its most basic form, $f(s) = \sum_{n=1}^{\infty} a_n e^{-s\lambda_n}$, s=$\sigma + i\omega$ where $\{\lambda_n\}$, n ≥ 1 is a sequence of all positive real numbers except perhaps λ_1, which can be positive or zero, strictly increasing to infinity and $\{a_n\}$, n=1,2,3.. is a sequence of complex numbers. As Apostol [1] notes, it was actually introduced for research in analytical number theory, specifically to establish the Prime Number Theory (PNT). Hardy [9] developed the general theory of Dirichlet series. However, the initial work was started by Bohr [10] who considered only the real values of the variable `s' and obtained many important results without any use of functional analysis techniques. The nature of the series' region of convergence and other results involving the complex values for "s" were also obtained.

However the systematic study of the growth, when f(s) is an entire function was first made by Ritt [11], who in fact, considered the Dirichlet series with positive exponents instead of the negative ones just to have an analogy with the study of growth of entire functions represented by Power series $\sum_{n=1}^{\infty} a_n z^n$.

Mendelbrojt in [12], has considered the Dirichlet series,

$$f(s) = \sum_{n=1}^{\infty} a_n e^{-s\lambda_n} \; \sigma + i\omega$$

where $\{\lambda_n\}$, n ≥ 1 is a sequence of all positive real numbers except perhaps λ_1, which can be positive or zero, strictly increasing to infinity and $\{a_n\}$, n=1,2,3.. is a sequence of complex numbers.

The series [1.2.1] is the general Dirichlet series with negative exponents. It is in fact, a generalized form of the Taylor series (Power series) given by

$$f(s) = \sum_{n=1}^{\infty} a_n z^n$$

The Taylor series [1.2.2], can be obtained from the Dirichlet series [1.2.1] by substituting z= $e^{-s}$and n = λ_n. The series [1.2.2] admits a radius of convergence R which may be equal to 0 ,∞ or to a positive finite number . If the Taylor series converges in $|z| < R$ then its radius of convergence is R given by the formula,

$$\frac{1}{R} = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}}$$

and $|z| = R$ is known as circle of convergence of the Taylor series [1.2.2] , in which the series converges absolutely .

Another generalized form of the Taylor series [1.2.2] is the Taylor-D series given by [1.2.3] below , which is obtained by substituting $z=e^{-s}$ in [1.2.2] i.e.

$$f(s) = \sum_{n=1}^{\infty} a_n e^{-sn}$$

About the Dirichlet series [1.2.1] , it has been shown by Apostol [1] that if the series converges for some s= $s_0 = \sigma_0 + i\omega_0$, then it converges at all those $s=\sigma + i\omega$ for which $\sigma > \sigma_0$ and the convergence is uniform in every closed angular domain given by

$$|\arg(s - s_0)| \geq \theta > \frac{\pi}{2}$$

The greatest lower bound (g.l.b.) of all these σ for which the series [1.2.1] converges is called the abscissa of ordinary convergence σ_c and is given by the formula

$$\sigma_c = \limsup_{n\to\infty} \frac{\log|\sum_{m=1}^{n} a_m|}{\lambda_n}$$

where $\sigma_c < \infty$ and does not converge if $\sigma < \sigma_c$.

The abscissa of absolute convergence σ_a is defined similarly and it is such that the series converges absolutely for $\sigma > \sigma_a$ and is given by ,

$$\sigma_a = \limsup_{n\to\infty} \frac{\log\sum_{m=1}^{n} |a_m|}{\lambda_n}$$

The difference between the values of σ_a and σ_c is estimated by the relation

$$0 \leq \sigma_a - \sigma_c \leq \limsup_{n\to\infty} \frac{\log n}{\lambda_n} = D$$

If D is zero then the abscissa of ordinary convergence i.e. σ_c and the abscissa of absolute convergence i.e.σ_a have the same value which is given by

$$\sigma_c = \sigma_a = \limsup_{k\to\infty} \frac{\log |\, a_k \,|}{\lambda_k}$$

It should be noted that the Dirichlet series and the Taylor-D series differ in the non-coincidence of their half planes of different kinds of convergences. It has also been proved earlier that a general Dirichlet series does not converge uniformly in every half plane $\sigma \geq \sigma_c$, although it does converge

uniformly in every closed bounded region of such a half plane. This is the case of the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{n^s}$ which does not converge uniformly in the half plane $\sigma > 1$ but converges uniformly in every closed bounded region of it . H.Bohr [10] introduced the notion of the abscissa of uniform convergence σ_u which is the greatest lower bound of quantities σ_0 such that the Dirichlet series converges uniformly for $\sigma > \sigma_0$. Bohr had established an inequality and showed that $\sigma_a \geq \sigma_u \geq \sigma_c$ where σ_u is given by

$$\sigma_u = \limsup_{n\to\infty} \frac{\log\left(\text{l.u.b.}\left|\sum_{m=1}^{n} a_m e^{it\lambda_m}\right|\right)}{\lambda_n}$$

The well known Cauchy formula for the co-efficients of the Taylor series was also extended by Bohr in [10]. Several results about functions analytic in a half plane represented by Dirichlet series or Generalized Laplace Transform can be seen in [2–4].

2 The class Ω(u,2)

In the present work, we shall throughout consider the functions `f' represented by Generalized Dirichlet series $f(s) = \sum_{k=1}^{\infty} a_k e^{s\lambda_k}$ with positive exponents rather than negative ones. It is just to have an analogy with the Power series $\sum_{k=1}^{\infty} a_k z^k$. The only difference in the characteristics of the Dirichlet series with positive exponents, as compared to the same, with negative exponents lies in the region of their convergence , that is what we call the half plane of convergence. The Dirichlet series with positive exponents converge in the left half plane whereas the one with negative exponents converge in the right half plane. We shall also take the liberty of interpreting the results of all those workers who have considered the Dirichlet series with negative exponents in our terminology.

It is well known that the Dirichlet series with positive exponents (frequencies) , in its most generalized form is given by

$$f(s) = \sum_{k=1}^{\infty} a_k e^{s\lambda_k}$$

where s=$\sigma + it\,(\sigma,\ trealvariables)$, $\{a_k\}$ in general, is a complex sequence , $\{\lambda_k\}$ is a strictly increasing sequence of positive real numbers and $\lambda_k \to \infty ask \to \infty$. It has been proved in [2] that the exponents $\{\lambda_k\}$ satisfy

$$\limsup_{k\to\infty} \frac{\log k}{\lambda_k} = D < \infty.$$

The abscissa of ordinary convergence (abs. convergence) $\sigma_c(\sigma_a)$ of the Dirichlet series [1.1] is defined as the least upper bound of all those `σ' for which the series converges (abs.). These are given as follows

$$\sigma_c = \begin{cases} -\limsup_{k\to\infty} \dfrac{\log\left|\sum_{m=1}^{k} a_m\right|}{\lambda_k} & provided \sigma \leq 0 or \sum_{k=1}^{\infty} a_k \text{ diverges} \\ -\limsup_{k\to\infty} \dfrac{\log\left|\sum_{m=k+1}^{\infty} a_m\right|}{\lambda_k} & provided \sigma > 0 or \sum_{k=1}^{\infty} a_k \text{ converges} \end{cases}$$

$$\sigma_a = \begin{cases} -\limsup_{k\to\infty} \dfrac{\log \mid \sum_{m=1}^{k} a_m \mid}{\lambda_k} & : \sigma \le 0 or \sum_{k=1}^{\infty} a_k \text{ diverges} \left(abs. \right) \\ -\limsup_{k\to\infty} \dfrac{\log \mid \sum_{m=k+1}^{\infty} a_m \mid}{\lambda_k} & : \sigma > 0 or \sum_{k=1}^{\infty} a_k \text{ converges} \left(abs. \right) \end{cases}$$

$$\le \sigma_c - \sigma_a \le D$$

If D given in [1.8.2] is zero then the abscissa of ordinary convergence i.e. σ_c and the abscissa of absolute convergence i.e.σ_a have the same value which is given by

$$\sigma_c = \sigma_a = -\limsup_{k\to\infty} \frac{\log \mid \alpha_k \mid}{\lambda_k}$$

$$= \sum_{k=1}^{\infty} \alpha_k e^{s\lambda_k}$$

be a fixed Dirichlet series with given exponents or frequencies $\{\lambda_k\}$ satisfying

$$\limsup_{k\to\infty} \frac{\log k}{\lambda_k} = D < \infty$$

and the coefficients $\{\alpha_k\}$ being a fixed sequence of non-zero complex numbers satisfying the condition

$$-\limsup_{k\to\infty} \frac{\log \mid \alpha_k \mid}{\lambda_k} = A$$

where A is any arbitrary but fixed real number. If D is zero, then the Dirichlet series u(s) has coincident abscissa of absolute convergence $\left(\sigma_a^u\right)$ and abscissa of ordinary convergence $\left(\sigma_c^u\right)$, which is given by the formula,

$$\sigma_c^u = \sigma_a^u = -\limsup_{k\to\infty} \frac{\log \mid \alpha_k \mid}{\lambda_k} = A$$

or equivalently by $-\limsup_{k\to\infty} |a_k|^{\frac{1}{\lambda_k}} = e^{-A}$.

The left plane $\sigma < A$ is denoted by R_u and is called the region of convergence of the fixed Dirichlet series u(s). Also, the series u(s) , converges uniformly in each left half plane σ= A-ϵ , $\epsilon > 0$. The sum of the series u(s) is an analytic function in the left half plane $\sigma < A$ and an entire function if A=+∞.The coefficients of the fixed Dirichlet series can be expressed in terms of its sum function by the Hadamard formula

$$\alpha_k = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} u\left(\sigma' + it'\right) e^{-\lambda_k (A' + it)} dt$$

where $A' < A$. Therefore, there can be no trivial expansion of the zero function and hence no other function can be expressed in the form of the series as given for u(s) in two ways. On the other hand, not every function can be represented by a Dirichlet series of the form [1.1] . For example, the function $e^{\beta z}$, $\beta \neq \lambda_k$, k=1,2,3... is not representable by a Dirichlet series .

Let the class Ω(u,2) is defined as follows ,

$\Omega(u,2)=\left\{f : f(s) = \sum_{k=1}^{\infty} a_k e^{s\lambda_k} : \sum_{k=1}^{\infty} \left|\frac{a_k}{\alpha_k}\right|^2 < \infty\right\}$

It can easily be seen that the class $\Omega(u,2)$ is well defined and the function `u' given by the fixed Dirichlet series u(s) does not belong to $\Omega(u,2)$.

Now, for $f,\ f(s) = \sum_{k=1}^{\infty} a_k e^{s\lambda_k}$ and g, $g(s)=\sum_{k=1}^{\infty} b_k e^{s\lambda_k} \in \Omega(u,\ 2)$

We define the following pointwise linear operations and norm in $\Omega(u,2)$ in the following way.

1. $(f+g)(s)=\sum_{k=1}^{\infty} (a_k+b_k)e^{s\lambda_k}$
2. $(\mu f)(s) = \sum_{k=1}^{\infty} \mu a_k e^{s\lambda_k}$ where $\mu is a scalar$
3. $\|f\|_2^2 = \sum_{k=1}^{\infty} \left|\frac{a_k}{\alpha_k}\right|^2$

Clearly, $\|f\|_2$ exists . It can further be seen that $\Omega(u,2)$ is a Normed Linear Space also.

3 Main results

The following theorem establishes the relation between the abscissa of convergence of u and f.

Theorem 1: Each element f of $\Omega(u,\ 2)$ has its abscissa of absolute convergence `$\sigma_a^f$' greater than or equal to the abscissa of absolute convergence `σ_a^u' of u .

Proof: We have ,

$\Omega(u,2) =\left\{f,\ f(s) = \sum_{k=1}^{\infty} a_k e^{s\lambda_k} \text{ where } \sum_{k=1}^{\infty} \left|\frac{a_k}{\alpha_k}\right|^2 < \infty\right\}$

Now, for any $f \in \Omega(u,\ 2)$ and each $k = 1,\ 2,\ 3$....we have

$$\left|\frac{a_k}{\alpha_k}\right|^2 < \mu^2 \text{ where } \mu \text{ is a positive constant}$$

$\Longrightarrow |a_k| < \mu\ |\ \alpha_k\ |$ for each k=1,2,3.....

$$\log\ |\ a_k\ |\ <\ \log\mu\ +\ \log\ |\ \alpha_k\ |$$

$$-\limsup_{k\to\infty} \frac{\log\ |\ a_k\ |}{\lambda_k} \geq -\limsup_{k\to\infty} \frac{\log\mu}{\lambda_k} - \limsup_{k\to\infty} \frac{\log\ |\ \alpha_k\ |}{\lambda_k}$$

$$\sigma_a^f \geq \sigma_a^u$$

Thus, each element f of $\Omega(u,\ 2)$ has its abscissa of absolute convergence `$\sigma_a^f$' greater than or equal to the abscissa of absolute convergence `σ_a^u' of u. Particularly , if $\sigma_a^u = \infty$ i.e. u is an entire function then $\sigma_a^f = \infty$ which implies `f' is also entire so in this case $\Omega(u,\ 2)$,consists of only entire functions but it does not include all entire functions for example $f(s) = \sum_{k=1}^{\infty} (-1)^k \alpha_k e^{s\log(\log k)}$ is an entire function but it does not belong to $\Omega(u,\ 2)$.

In terms of convergence regions (typically a right half-plane $\mathrm{Re}\,(s) > \sigma_a$ in the theory of Dirichlet series , this inequality means that the half-plane of absolute convergence of u contains (or is at least as large as) the half-plane of absolute convergence of every $f \in \Omega(u,\ 2)$:

$$\{s \in \mathrm{C} : \mathrm{Re}\,(s) > \sigma_a^u\} \subseteq \{s \in \mathrm{C} : \mathrm{Re}\,(s) > \sigma_a^f\} \text{for all } f \in \Omega(u,\ 2)$$

Equivalently, σ_a^u serves as a **lower bound** on the abscissae of absolute convergence across the entire family $\Omega(u,\ 2)$, meaning no element $f \in \Omega(u,\ 2)$ can converge absolutely strictly further to the left than u does.

Next, we have proved that Ω(u,2) is a Banach space which becomes a Hilbert space and possess the property of Uniform convergence [3] over every compact subset in the region of convergence R_u of u .

THEOREM 2: Ω(u, 2) is a Banach space.

PROOF: Let $\{f_l\}$ be a Cauchy sequence in Ω(u,2) such that $f_l(s) = \sum_{k=1}^{\infty} a_{lk} e^{s\lambda_k}$ then

$$\sum_{\mathrm{k}=1}^{\infty} \left| \frac{\mathrm{a_{lk}}}{\alpha_{\mathrm{k}}} \right|^2 < \infty$$

Since,$\{f_l\}$ is a Cauchy sequence in Ω(u,2) so, for a given $\in > 0$, there exists a positive integer $\mathrm{N}_0(\in)$ such that

$\|f_l - f_m\|_2 < \in$ for l,m> $\mathrm{N}_0(\in)$

$$\left(\sum_{k=1}^{\infty} \left| \frac{a_{lk} - a_{mk}}{\alpha_k} \right|^2 \right)^{\frac{1}{2}} < \in$$

$$\sum_{k=1}^{\infty} \left| \frac{a_{lk} - a_{mk}}{\alpha_k} \right|^2 < \in^2$$

$$\Longrightarrow \left| \frac{a_{lk} - a_{mk}}{\alpha_k} \right|^2 < \in^2 \text{ for each k}$$

$$\Rightarrow \left| \frac{a_{lk} - a_{mk}}{\alpha_k} \right| < \in \text{ for each k}$$

This shows that $\left\{ \frac{a_{lk}}{\alpha_k} \right\}$ is a Cauchy sequence in C for each k and so converges to $\frac{a_{0k}}{\alpha_k}$ (say) as $l \to \infty$.

i.e. $\lim_{l \to \infty} \frac{a_{lk}}{\alpha_k} = \frac{a_{0k}}{\alpha_k}$

Define a $f_0(s) = \sum_{k=1}^{\infty} a_{0k} e^{s\lambda_k}$, we show that $f_l \to f_0$ as $l \to \infty$ and $f_0 \in \Omega(\mathrm{u},\ 2)$.

From [3.5.1], we see that for any positive integer m, $\sum_{k=1}^{m} \left| \frac{a_{lk} - a_{mk}}{\alpha_k} \right|^2 < \in^2$. Let $\mathrm{m} \to \infty$ then $\sum_{k=1}^{\infty} \left| \frac{a_{lk} - a_{0k}}{\alpha_k} \right|^2 < \in^2 < \in$ for $l > N_0(\in)$. This shows that $(f_l - f_0)$ belongs to Ω(u,2) such that $\|f_l - f_0\|_2 < \in$ i.e. $f_l \to f_0$ as $l \to \infty$. Now it follows then $f_0 = f_l + (f_0 - f_l) \in \Omega(u,\ 2)$.

For any $\varepsilon > 0$, there exists a sufficiently large index l such that:

$$\|f_l - f_0\|_2 < \varepsilon$$

This shows that $f_l \to f_0$ in the $\|\cdot\|_2$-metric norm as $l \to \infty$, and the difference $f_l - f_0$ (as well as $f_0 - f_l = -(f_l - f_0)$) lies in $\Omega(u,\ 2)$. That is $f_l \in \Omega(u,\ 2)$ and $(f_0 - f_l) \in \Omega(u,\ 2) \Longrightarrow f_0 = f_l + (f_0 - f_l) \in \Omega(u,\ 2)$

The topological significance holds in the fact that Since an arbitrary convergent sequence $\{f_l\}$ in $\Omega(u,\ 2)$ with limit f_0 satisfies $f_0 \in \Omega(u,\ 2)$, the set $\Omega(u,\ 2)$ is closed in the topology induced by $\|\cdot\|_2$

(making it a complete subspace whenever the ambient space under $\|\cdot\|_2$ is complete). Also, decomposing f_0 into a known element f_l plus a small perturbation $f_0 - f_l$ lets to pass structural or growth properties from f_l and the small error term directly to the limit function f_0. It's also noteworthy that once $f_0 \in \Omega(u,\ 2)$ is secured with the characteristics it holds, any general property established for all members of $\Omega(u,\ 2)$ such as $\sigma_a^{f_0} \geq \sigma_a^u$ automatically applies to the limit function f_0 as well.

Uniform convergence on every compact subset of R_u means the map $f \mapsto f(s)$ is continuous into the space of analytic/holomorphic functions endowed with the topology of uniform convergence on compact sets (compact-open topology) over R_u.

In complex analysis—especially for Dirichlet series, generalized Dirichlet series—the next theorem bridges abstract space convergence ($f_l \to f$ in $\Omega(u,\ 2)$) with pointwise/uniform function convergence on compact sets inside R_u.

THEOREM 3 : The convergence in the topology of $\Omega(u,\ 2)$ implies uniform convergence of the corresponding functions on every compact subset $K \subset R_u$, where R_u denotes the region of convergence of the reference element u. Explicitly $f_l \to f in \Omega(u,\ 2) \Longrightarrow f_l(s) \to f(s)$ uniformly over every compact subset of the region of convergence R_u of u.

PROOF : Let S be a compact subset in the region of convergence R_u, then we get a rectangle T in R_u containing S where

$$\{(\sigma,\ t) : \sigma_1 \leq \sigma \leq \sigma_2,\ t_1 \leq t \leq t_2\}$$

Let σ_3 satisfies $\sigma_2 < \sigma_3 < \sigma$ so that $\theta = \dfrac{e^{\sigma_2}}{e^{\sigma_3}} < 1$.Then for any given $\epsilon > 0$ choose a `η' such that

$\eta.M\sum_{k=1}^{\infty} \theta^{\lambda_k} < \epsilon$ where $|\alpha_k| e^{\lambda_k \sigma_3} \leq M < \infty$

and $\|f_l - f\|_2 < \eta$ for $l \geq l_0(\eta)$

$\Longrightarrow \sum_{k=1}^{\infty} \left| \dfrac{a_{lk} - a_k}{\alpha_k} \right|^2 \leq \eta^2$ for $l \geq l_0(\eta)$

$\left| \dfrac{a_{lk} - a_k}{\alpha_k} \right|^2 < \eta^2$ for each k=1,2,3,.... $l \geq l_0(\eta)$

Then for s $\in S$ and $l \geq l_0$, we have

$$|f_l(s) - f(s)| \leq \sum_{k=1}^{\infty} |a_{lk} - a_k| e^{\sigma \lambda_k}$$

$$\leq \sum_{k=1}^{\infty} \left| \frac{a_{lk} - a_k}{\alpha_k} \right| |\alpha_k| e^{\sigma \lambda_k}$$

$$\leq \eta \sum_{k=1}^{\infty} |\alpha_k| e^{\sigma_3 \lambda_k} \leq \eta M \sum_{k=1}^{\infty} \theta^{\lambda_k} < \epsilon$$

which proves uniform convergence over every compact subset of the region of convergence R_u of u. The sequence $\{f_l\}$ converges to f in the topology or norm defining $\Omega(u,\ 2)$ (such as the $\|\cdot\|_2$-norm or seminorm family generating $\Omega(u,\ 2)$). This controls the coefficients, growth, or tail behavior of $f_l - f$ globally or with respect to the weight/structure of u. For R_u being the convergence region of u (for instance, a half-plane $\mathrm{Re}\,(s) > \sigma_0^u$ or $\mathrm{Re}\,(s) > \sigma_a^u$), any compact subset $K \subset R_u$ is closed and bounded inside R_u, so it stays safely away from any boundary/convergence threshold—for example, on a half-plane $\mathrm{Re}\,(s) > \sigma_u$, every compact $K \subset R_u$ satisfies:

$$\mathrm{Re}\,(s) \le M_K \text{and}\, \mathrm{Re}\,(s) \ge \sigma_u + \delta_K (\delta_K > 0)$$

for all $s \in K$.

Also, For every compact set $K \subset R_u$ and every $\varepsilon > 0$, there exists an index $L = L(K, \varepsilon)$ (independent of $s \in K$) such that for all $l \ge L$:

$$\sup_{s \in K} |f_l(s) - f(s)| < \varepsilon$$

It is also important to note that as the uniform limits of analytic functions on compact sets remain analytic on R_u, any limit function f inherits analyticity on R_u, and derivatives $f_l^{(m)}(s)$ also converge uniformly to $f^{(m)}(s)$ on compact subsets of the interior of R_u (by Weierstrass's / Cauchy's estimates). Additionally, on a compact subset $K \subset R_u$ prevents boundary blow-up or slow boundary behavior, allowing the $\|\cdot\|_2$-convergence of $f_l - f$ to dominate $|f_l(s) - f(s)|$ uniformly across all $s \in K$.

The following theorem connects normed space geometry with inner product (Hilbert space) structure on $\Omega(u, 2)$ via the Jordan–von Neumann theorem (parallelogram characterization of inner-product norms). An inner product space is a Hilbert space if and only if it is complete with respect to its induced norm (every Cauchy sequence converges in $\Omega(u, 2)$). Thus, once completeness (Banach property) of $\Omega(u, 2)$ is established, the parallelogram law upgrades $\Omega(u, 2)$ directly into a Hilbert space—giving you orthogonal projections, Riesz representation, and orthonormal expansions in that space. Specifically, if $\Omega(u, 2)$ is already a Banach space (i.e., a complete normed linear space under its norm $\|\cdot\|_2$), and norm $\|.\|_2$ satisfies the parallelogram law then $\Omega(u, 2)$ is a Hilbert space. many specific results about Hilbert space can be found in [5–8].

THEOREM 4: The norm defined on $\Omega(u, 2)$ *satisfies parallelogram law equivalently* $\Omega(u, 2)$ becomes Hilbert space.

PROOF : Let $f(s) = \alpha_1 e^{s\lambda_1} + \alpha_2 e^{s\lambda_2}$, $g(s) = \alpha_1 e^{s\lambda_1} + (-\alpha_2) e^{s\lambda_2}$.Clearly, $f, g \in \Omega(u, 2)$.

$\|f\|_2^2 = \sum_{k=1}^{2} \left|\frac{a_k}{\alpha_k}\right|^2 = \left|\frac{\alpha_1}{\alpha_1}\right|^2 + \left|\frac{\alpha_2}{\alpha_2}\right|^2 \Rightarrow \|f\|_2 = 2^{\frac{1}{2}}$, similarly $\|g\|_2 = 2^{\frac{1}{2}}$

Also $\|f + g\|_2^2 = \left|\frac{2\alpha_1}{\alpha_1}\right|^2 = 2^2 \Longrightarrow \|f + g\|_2 = 2$

and $\|f - g\|_2^2 = \left|\frac{2\alpha_2}{\alpha_2}\right|^2 = 2^2 \Longrightarrow \|f - g\|_2 = 2$

Clearly $\|f + g\|_2^2 + \|f - g\|_2^2 = 2.\|f\|_2^2 + 2.\|g\|_2^2$

Thus, Parallelogram law holds showing that it is a Hilbert space.

4 Conclusion

Conclusion: The results in this paper establish a coherent structural and analytical framework for the class $\Omega(u, 2)$ in relation to the fixed Dirichlet series u and its region of convergence R_u. Every member $f \in \Omega(u, 2)$ has an abscissa of absolute convergence bounded below by that of $u(\sigma_a^f \ge \sigma_a^u)$, ensuring that σ_a^u anchors a lower convergence threshold across the entire family while norm convergence within $\Omega(u, 2)$ combined with closure under addition guarantees that limits of the form $f_0 = f_l + (f_0 - f_l)$ remain inside $\Omega(u, 2)$. Moreover, convergence in $\Omega(u, 2)$ carries over to uniform convergence on every compact subset of R_u, linking abstract space topology with concrete analytic

behavior on compact regions, while fulfillment of the parallelogram law (together with completeness) equips $\Omega(u, 2)$ with a Hilbert space structure. Consequently, $\Omega(u, 2)$ provides a stable, topologically well-behaved, and inner-product-equipped setting for analyzing convergence and growth properties governed by u.

References

[1] T. M. Apostol. *Introduction to Analytic Number Theory*. Undergraduate Texts in Mathematics, Springer-Verlag, New York, 1976 (reprinted 1980).

[2] K. Chauhan. "On a class of functions analytic in a half plane represented by Dirichlet series." *J.P.A.S.*, Vol. 17, pp. 30–39, 2011.

[3] K. Chauhan, "A Banach algebra of functions analytic in a half plane", *Journal of Chemical Biological and Physical Sciences*, 4(2): 1437–1443, April 2014.

[4] B.L. Srivastava, K. Chauhan, "On the convergence of generalized Laplace integral", *The Journal of Analysis*, Vol 13,73-82, January 2005.

[5] H. Hedenmalm, P. Lindqvist, and K. Seip. "A Hilbert space of Dirichlet series and the system of dilations." *Duke Mathematical Journal*, Vol. 86, No. 1, pp. 1–37, 1997.

[6] J. Weidmann. *Linear Operators in Hilbert Spaces*. Graduate Texts in Mathematics, Vol. 68, Springer-Verlag, New York, 1980.

[7] N. Kumar and G. Manocha. "On a class of entire functions represented by Dirichlet series." *Journal of the Egyptian Mathematical Society*, Vol. 21, No. 1, pp. 21–24, 2013.

[8] A. Bondarenko, O. F. Brevig, E. Saksman, and K. Seip. "Linear space properties of spaces of Dirichlet series." *Transactions of the American Mathematical Society*, Vol. 372, No. 9, pp. 6677–6702, 2019.

[9] G. H. Hardy and M. Riesz. *The General Theory of Dirichlet's Series*. Cambridge Tracts in Mathematics and Mathematical Physics, No. 18, Cambridge University Press, Cambridge, 1915.

[10] H. Bohr. "A theorem concerning power series." *Proceedings of the London Mathematical Society*, Series 2, Vol. 13, pp. 1–5, 1914.

[11] J. F. Ritt. "Note on the Dirichlet series with complex exponents." *Annals of Mathematics*, Second Series, Vol. 26, No. 1/2, pp. 75–78, 1924.

[12] S. *Mandelbrojt*. Dirichlet Series: Principles and Methods. D. Reidel Publishing Company, Dordrecht / Springer-Verlag, 1972.